COMPENDIUM OF CLAIMS AND PROOFS, PART I

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Contents

1. The Star Product

Definition 1.1 (The Star Operator). ***(October 17, 2010, January 17, 2013)***

- ***On Two Surfaces***

Let $f(x,y)$ and $g(x,y)$ be surfaces so that $f,g\colon [0,1]\times[0,1]\to\mathbb{R}$. The star operator $\star\colon [0,1]^2\times[0,1]^2\to[0,1]^2$ takes two surfaces and creates another in the following way:

$$(f(x,y),g(x,y)) \rightsquigarrow (f(1-y,z),g(x,y)) \rightsquigarrow h(x,z) \rightsquigarrow h(x,y)$$

with the central transformation being defined by $\diamond\colon [0,1]^2\times[0,1]^2\to[0,1]^2$

$$f(1-y,z)\diamond g(x,y)=\int_0^1 f(1-y,z)g(x,y)\,dy = h(x,z)$$

and the last transformation that takes $h(x,z)\rightsquigarrow h(x,y)$ we will call $j\colon [0,1]^2\to[0,1]^2$. Thus

$$\boxed{f(x,y)\star g(x,y) = j\left(f(1-y,z)\diamond g(x,y)\right) = j\left(\int_0^1 f(1-y,z)g(x,y)\,dy\right)}$$

In the special case where $f(x,0)$, we have

$$f(1-y,0)\diamond g(x,y)=\int_0^1 f(1-y,0)g(x,y)\,dy=h(x,0)$$

motivating the following simplification:

- ***On a Function and a Surface***

Let $f(x)$ be a function $f\colon[0,1]\to\mathbb{R}$, and $g(x,y)$ a function such that $g\colon[0,1]^2\to\mathbb{R}$. The star operator $\star\colon[0,1]\times[0,1]^2\to[0,1]$ takes the function and the surface and creates another function in the following way:

$$(f(x),g(x,y))\rightsquigarrow(f(1-y),g(x,y))\rightsquigarrow h(x)$$

with the last transformation being defined by $\diamond\colon[0,1]\times[0,1]^2\to[0,1]$

$$f(1-y)\diamond g(x,y)=\int_0^1 f(1-y)g(x,y)\,dy=h(x)$$

Thus we have

$$\boxed{f(x)\star g(x,y)=f(1-y)\diamond g(x,y)=\int_0^1 f(1-y)g(x,y)\,dy}$$

1.1. Properties of the Diamond Operator.

1.1.1. *Linearity.*

Claim 1.1. ***(January 29, 2013)*** *The diamond operator is a linear operator.*

Proof Linearity follows from the integral operator properties. Thus we have:

- ***Surface diamond Surface***

Letting $f,g,h\colon[0,1]^2\to\mathbb{R}$, and c be a constant,

$$\begin{aligned}(c\cdot f)\diamond g&=\int_0^1(c\cdot f(x,y))\,g(x,y)\,dy\\&=c\int_0^1 f(x,y)g(x,y)\,dy\\&=c\cdot(f\diamond g)\end{aligned}$$

$$\begin{aligned}f\diamond(c\cdot g)&=\int_0^1 f(x,y)\,(c\cdot g(x,y))\,\,dy\\&=c\int_0^1 f(x,y)g(x,y)\,dy\\&=c\cdot(f\diamond g)\end{aligned}$$

and

$$\begin{aligned}(f+g)\diamond h&=\int_0^1(f(x,y)+g(x,y))\,h(x,y)\,dy\\&=\int_0^1 f(x,y)h(x,y)\,dy+\int_0^1 g(x,y)h(x,y)\,dy\\&=f\diamond h+g\diamond h\end{aligned}$$

$$\begin{aligned}f\diamond(g+h)&=\int_0^1 f(x,y)\,(g(x,y)+h(x,y))\,\,dy\\&=\int_0^1 f(x,y)g(x,y)\,dy+\int_0^1 f(x,y)h(x,y)\,dy\\&=f\diamond g+f\diamond h\end{aligned}$$

- ***Function diamond Surface***

Letting $d, e\colon [0,1] \to \mathbb{R}$ and $f, g\colon [0,1]^2 \to \mathbb{R}$, and c be a constant,

$$\begin{aligned}
(c \cdot d) \diamond f &= \int_0^1 (c \cdot d(x))\, f(x,y)\, dy \\
&= c \int_0^1 d(x) f(x,y)\, dy \\
&= c \cdot (d \diamond f) \\
d \diamond (c \cdot f) &= \int_0^1 d(x)\, (c \cdot f(x,y))\ dy \\
&= c \int_0^1 d(x) f(x,y)\, dy \\
&= c \cdot (d \diamond f)
\end{aligned}$$

and

$$\begin{aligned}
(d+e) \diamond f &= \int_0^1 (d(x) + e(x))\, f(x,y)\, dy \\
&= \int_0^1 d(x) f(x,y)\, dy + \int_0^1 e(x) f(x,y)\, dy \\
&= d \diamond f + e \diamond f \\
d \diamond (f+g) &= \int_0^1 d(x)\, (f(x,y) + g(x,y))\ dy \\
&= \int_0^1 d(x) f(x,y)\, dy + \int_0^1 d(x) g(x,y)\, dy \\
&= d \diamond f + d \diamond g
\end{aligned}$$

□

1.2. **Properties of the J Transformation.**

1.2.1. *Linearity.*

Claim 1.2. ***(January 29, 2013)*** *The transformation $j\colon [0,1]^2 \to [0,1]^2$ is likewise linear.*

Proof Let $f, g\colon [0,1]^2 \to \mathbb{R}$, and c be a constant. Then:

$$\begin{aligned}
j\,(c \cdot f(x,z)) &= c \cdot f(x,y) \\
&= c \cdot j\,(f(x,z))
\end{aligned}$$

Also:

$$\begin{aligned}
j\,(f(x,z) + g(x,z)) &= f(x,y) + g(x,y) \\
&= j\,(f(x,z)) + j\,(g(x,z))
\end{aligned}$$

□

1.2.2. *Other Properties.*

Claim 1.3. ***(January 17, 2013)*** *Take the transformation $j\colon [0,1]^2 \to [0,1]^2$ that carries $h(x,z) \rightsquigarrow h(x,y)$. Then:*

$$\int_a^b j(h(x,z))\, dx = j\left(\int_a^b h(x,z)\, dx\right)$$

Proof

$$\int_a^b j(h(x,z))\, dx = \int_a^b h(x,y)\, dx = H(0,y)$$

On the other hand

$$j\left(\int_a^b h(x,z)\, dx\right) = j\,(H(0,z)) = H(0,y)$$

□

1.3. **Properties of the Star Operator.**

1.3.1. *Linearity.*

Corollary 1.4 (Scaling of the Star Product). ***(January 30, 2013)***
Let $d(x)$ be a function $d\colon [0,1] \to \mathbb{R}$, and $f(x,y)$ and $g(x,y)$ be functions $f,g\colon [0,1]^2 \to \mathbb{R}$, c is a constant. Then:

- **Surface star Surface**

$$(c \cdot f) \star g = c \cdot (f \star g)$$

and

$$f \star (c \cdot g) = c \cdot (f \star g)$$

- **Function star Surface**

$$(c \cdot d) \star f = c \cdot (d \star f)$$

and

$$d \star (c \cdot f) = c \cdot (d \star f)$$

Proof This is a consequence of **Claim 1.1** and **Claim 1.2**.

- ***Surface star Surface***

$$\begin{aligned}(c \cdot f) \star g &= j\left((c \cdot f(1-y,z)) \diamond g(x,y)\right)\\ &= c \cdot j\left(f(1-y,z) \diamond g(x,y)\right)\\ &= c \cdot (f \star g)\end{aligned}$$

and

$$\begin{aligned}f \star (c \cdot g) &= j\left(f(1-y,z) \diamond (c \cdot g(x,y))\right)\\ &= c \cdot j\left(f(1-y,z) \diamond g(x,y)\right)\\ &= c \cdot (f \star g)\end{aligned}$$

- ***Function star Surface***

$$\begin{aligned}(c \cdot d) \star f &= j\left((c \cdot d(1-y)) \diamond f(x,y)\right)\\ &= c \cdot j\left(d(1-y) \diamond f(x,y)\right)\\ &= c \cdot (d \star f)\end{aligned}$$

and

$$\begin{aligned}d \star (c \cdot f) &= j\left(d(1-y) \diamond (c \cdot f(x,y))\right)\\ &= c \cdot j\left(d(1-y) \diamond f(x,y)\right)\\ &= c \cdot (d \star f)\end{aligned}$$

□

Corollary 1.5 (Distributive Property of the Star Product). ***(January 27, 2013)***
Let $d(x), e(x)$ be a functions $d, e\colon [0,1] \to \mathbb{R}$ and $f(x,y), g(x,y)$ and $h(x,y)$ be functions $f,g,h\colon [0,1]^2 \to \mathbb{R}$, c is a constant. Then:

- **Surface star Surface**

$$(f+g) \star h = f \star h + g \star h$$

and

$$f \star (g+h) = f \star g + f \star h$$

- **Function star Surface**

$$(d+e) \star f = d \star f + e \star f$$

and

$$d \star (f+g) = d \star f + d \star g$$

Proof Again this is a consequence of **Claim 1.1** and **Claim 1.2**.

- ***Surface star Surface***

We have:

$$\begin{aligned}(f+g)\star h &= j\left((f(1-y,z)+g(1-y,z))\diamond h(x,y)\right)\\ &= j\left(f(1-y,z)\diamond h(x,y)+g(1-y,z)\diamond h(x,y)\right)\\ &= j\left(f(1-y,z)\diamond h(x,y)\right)+j\left(g(1-y,z)\diamond h(x,y)\right)\\ &= f\star h+g\star h\end{aligned}$$

The second to last line is again a consequence of **Claim 1.2**.

The second part is:

$$\begin{aligned}f\star(g+h) &= j\left(f(1-y,z)\diamond(g(x,y)+h(x,y))\right)\\ &= j\left(f(1-y,z)\diamond g(x,y)+f(1-y,z)\diamond h(x,y)\right)\\ &= j\left(f(1-y,z)\diamond g(x,y)\right)+j\left(f(1-y,z)\diamond h(x,y)\right)\\ &= f\star g+f\star h\end{aligned}$$

where the second to last line is justified by **Claim 1.2**.

- ***Function star Surface***

Rewriting the first part,

$$\begin{aligned}(d+e)\star f &= (d(1-y)+e(1-y))\diamond f(x,y)\\ &= d(1-y)\diamond f(x,y)+e(1-y)\diamond f(x,y)\\ &= d\star f+e\star f\end{aligned}$$

follows directly from the linear properties of the diamond operator **Claim 1.1**.

Next, in the second part,

$$\begin{aligned}d\star(f+g) &= d(1-y)\diamond(f(x,y)+g(x,y))\\ &= d(1-y)\diamond f(x,y)+d(1-y)\diamond g(x,y)\\ &= d\star f+d\star g\end{aligned}$$

again by **Claim 1.1**. □

1.3.2. *Other Properties.*

Claim 1.6 (Zero-property). ***(March 31, 2013)*** *Let f be a surface, with $f\colon[0,1]^2\to\mathbb{R}$. We have that*

$$0\star f=0$$

The statement

$$f\star 0=0$$

is also true if $f\colon[0,1]\to\mathbb{R}$, f is a function.

Proof First,

$$\begin{aligned}0\star f &= \int_0^1 0\cdot f(x,y)\,dy\\ &= \int_0^1 0\,dy\\ &= 0\end{aligned}$$

Next, with $f\colon[0,1]^2\to\mathbb{R}$,

$$\begin{aligned}f\star 0 &= \int_0^1 f(1-y,z)\cdot 0\,dy\\ &= \int_0^1 0\,dy\\ &= 0\end{aligned}$$

and we can see that if $f\colon[0,1]\to\mathbb{R}$ we get

$$\begin{aligned}f\star 0 &= \int_0^1 f(1-y)\cdot 0\,dy\\ &= \int_0^1 0\,dy\\ &= 0\end{aligned}$$

□

Claim 1.7 (Non-commutativity). ***(October 17, 2010)*** *The star product is non-commutative.*

Proof by Counterexample The claim only makes sense from the viewpoint of surfaces.

- ***On Surfaces***

Let $f, g\colon [0,1]^2 \to \mathbb{R}$.

We want to show that $f(x,y) \star g(x,y) \neq g(x,y) \star f(x,y)$. Choose $f(x,y) = x$ and $g(x,y) = y$ without loss of generality. Then:

$$f(x,y) \star g(x,y) = f(x) \star g(x,y) = \int_0^1 f(1-y) \cdot g(x,y)\, dy = \int_0^1 (1-y) \cdot y\, dy = \int_0^1 \left(y - y^2\right)\, dy = \frac{1}{6}$$

and

$$g(x,y) \star f(x,y) = j\left(\int_0^1 g(1-y,z) \cdot f(x,y)\, dy\right) = j\left(\int_0^1 (z \cdot x)\, dy\right) = j\,(z \cdot x) = y \cdot x$$

are non-equal. □

Claim 1.8 (Associativity). ***(October 17, 2010)*** *The star product is associative.*

Proof Again the claim only makes sense for the star operator on surfaces.

- ***On Surfaces***

Let $f, g, h\colon [0,1]^2 \to \mathbb{R}$.

We want to show that $[f(x,y) \star g(x,y)] \star h(x,y) = f(x,y) \star [g(x,y) \star h(x,y)]$.

The LHS is:

$$\begin{aligned}[f(x,y) \star g(x,y)] \star h(x,y) &= j\left(\int_0^1 f(1-y,z) g(x,y)\, dy\right) \star h(x,y) \\ &= j\left(\int_0^1 \int_0^1 f(1-z,w) g(1-y,z)\, dz\, h(x,y)\, dy\right) \\ &= j\left(\int_0^1 \int_0^1 f(1-z,w) g(1-y,z)\, h(x,y)\, dz\, dy\right)\end{aligned}$$

The RHS is:

$$\begin{aligned}f(x,y) \star [g(x,y) \star h(x,y)] &= f(x,y) \star j\left(\int_0^1 g(1-y,z) h(x,y)\, dy\right) \\ &= j\left(\int_0^1 f(1-z,w) \int_0^1 g(1-y,z) h(x,y)\, dy\, dz\right) \\ &= j\left(\int_0^1 \int_0^1 f(1-z,w) g(1-y,z) h(x,y)\, dy\, dz\right)\end{aligned}$$

We immediately see the equivalence using the Fubini Theorem to exchange the order of integration. □

1.4. Mechanics of Powering.

Claim 1.9 (Powering Symmetry). ***(December 30, 2012)*** *Take $f\colon [0,1]^2 \to \mathbb{R}$. Let the nth power of $f(x,y)$, $f_n(x,y)$ be denoted shorthand by F^n (for* Pasquali patches*, we use the notation P^n and $p_n(x,y)$ interchangeably). Then for $m, n \geq 1$, $F^n \star F^m = F^m \star F^n$.*

Proof by Induction By definition of powering,

$$F^1 \star F^m = F^1 \star \underbrace{(F^1 \star \ldots \star F^1)}_{\text{m times}}$$

Then, by **Claim 1.8 Associativity of the Star Product**, this is equivalent to

$$\underbrace{(F^1 \star \ldots \star F^1)}_{\text{m times}} \star F^1 = F^m \star F^1$$

Next assume that, for fixed $k, m \geq 1$, $F^k \star F^m = F^m \star F^k$.

Then $F^{k+1} \star F^m = (F^1 \star F^k) \star F^m$, where this bit we take as the definition of powering, and then, by **Claim 1.8** again, $F^1 \star (F^k \star F^m)$, which then by our inductive hypothesis equals $F^1 \star (F^m \star F^k)$ and this is $(F^m \star F^k) \star F^1$

by the inductive basis. Again, **Claim 1.8** gives $F^m \star (F^k \star F^1)$ and lastly $F^m \star F^{1+k}$ by definition of powering. Axiomatic commutativity of the positive integers gives $F^m \star F^{k+1}$ and we are done.

By symmetry of equality itself, we need only do a single proof of induction on one of the power parameters. □

Claim 1.10. ***(December 30, 2012)*** *Again take* $f\colon [0,1]^2 \to \mathbb{R}$. $F^n \star F^m = F^{m+n}$.

Proof by Induction First, fix $m \geq 1$. By the definition of powering, $F^1 \star F^m = F^{m+1}$. Next, suppose it's true that $F^k \star F^m = F^{m+k}$. Then $F^{k+1} \star F^m$ equals $(F^1 \star F^k) \star F^m$ by definition of powering, and by **Claim 1.8 Associativity of the Star Product** we get $F^1 \star (F^k \star F^m)$ or $F^1 \star F^{m+k}$ using the inductive hypothesis. By the inductive basis, this becomes $F^{(m+k)+1}$, that is $F^{m+(k+1)}$ using associativity of the positive integers, which we take as axiomatic.

Having done so, next fix $n \geq 1$ and let m vary. We have $F^n \star F^1 = F^1 \star F^n$ by **Claim 1.9 Powering Symmetry**. Since this part of the proof is identical to the one we just wrote, we are done. □

2. *Pasquali Patches*

2.1. Definitions and Constructions.

Definition 2.1 (*Pasquali patch*)**.** ***(April 22, 2010)*** *Define a continuous surface* $p(x,y)$*, with* $p\colon [0,1] \times [0,1] \to \mathbb{R}^+ \cup \{0\}$*, and let* $\int_0^1 p(x,y)\,dx = 1$ *be true regardless of the value of* y*. In other words, integrating such surface with respect to* x *yields the uniform probability distribution* $u(y)$*,* $u\colon [0,1] \to \{1\}$*. We will call this a* Pasquali patch.

Construction 2.1. ***(October 10, 2010)*** *A way to construct a* Pasquali patch *is by positing*

$$p(x,y) = f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)\int_0^1 f_1(x)\,dx}{\int_0^1 f_2(x)\,dx} = f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)F_1}{F_2}$$

for arbitrary function choices $f_{1,2}(x)$*, and* $g_1(y)$*.*

Proof Since for *Pasquali patches* $\int_0^1 p(x,y)\,dx = 1$, it can be seen that such construction is a *Pasquali patch* because

$$\int_0^1 \left(f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)F_1}{F_2} \right) dx = g_1(y)F_1 + \frac{F_2}{F_2} - \frac{g_1(y)F_1F_2}{F_2} = 1$$

regardless of function choices of y. □

2.2. Properties of *Pasquali Patches* Under the Star Operator.

2.2.1. *Notable Properties.*

Claim 2.1 (Closure of *Pasquali Patches*)**.** ***(October 12, 2010)*** *A* Pasquali patch *star a* Pasquali patch *yields a new* Pasquali patch*. In particular, a* Pasquali patch *star itself (*Pasquali patch *powers) will always yield another* Pasquali patch.

Proof Take *Pasquali patches* $p(x,y)$ and $q(x,y)$. We want to show that $p(x,y) \star q(x,y) = r(x,y)$ is a *Pasquali patch*.

Thus, we want to show that $\int_0^1 r(x,y)\,dx = 1$. In other words,

$$\int_0^1 j\left(\int_0^1 p(1-y,z)q(x,y)\,dy \right) dx = 1$$

By **Claim 1.3**, we can write this as

$$j\left(\int_0^1 \int_0^1 p(1-y,z)q(x,y)\,dy\,dx \right)$$

We can exchange the order of integration because of absolute convergence of the integrals (Fubini Theorem). Thus

$$j\left(\int_0^1 p(1-y,z) \int_0^1 q(x,y)\,dx\,dy \right)$$

yields

$$j\left(\int_0^1 p(1-y,z)u(y)\,dy \right) = j\left(\int_0^1 p(1-y,z)\,dy \right)$$

This final integral evaluates to $j\left(u(z)\right) = u(y) = 1$ for any choice of y. Thus $\int_0^1 r(x,y)\,dx = 1$. □

Example 2.1. ***(October 12, 2010)*** *Take the* Pasquali patch $p(x,y) = x^2y^3 + x\left(2 - \frac{2y^3}{3}\right)$*. It is evident it is a* Pasquali patch *because*

$$\int_0^1 \left[x^2y^3 + x\left(2 - \frac{2y^3}{3}\right)\right] dx = \left[\frac{x^3}{3}y^3 + \frac{x^2}{2}\left(2 - \frac{2y^3}{3}\right)\right]\Bigg|_0^1 = 1$$

We can calculate

$$p(1-y,z) = z^3y^2 - \frac{4z^3y}{3} + \frac{z^3}{3} - 2y + 2$$

so that the second "power" of the Pasquali patch *is*

$$p_2(x,y) = j\left(\int_0^1 p(1-y,z)p(x,y)\,dy\right) = j\left(\frac{29x}{15} + \frac{z^3x}{90} + \frac{x^2}{10} - \frac{z^3x^2}{60}\right)$$

and the final transformation j of the star operator gives

$$p_2(x,y) = \frac{29x}{15} + \frac{y^3x}{90} + \frac{x^2}{10} - \frac{y^3x^2}{60}$$

One easily checks $p_2(x,y)$ is indeed a Pasquali patch *by performing the integral $\int_0^1 p_2(x,y)\,dx$ and ascertaining its equality to* 1*. The third* Pasquali patch *power is*

$$p_3(x,y) = \frac{1741x}{900} - \frac{y^3x}{5400} + \frac{59x^2}{600} + \frac{y^3x^2}{3600}$$

Example 2.2. ***(October 15, 2010, October 17, 2010)*** *The* Pasquali patch

$$p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$$

has powers:

$$\begin{aligned}
p(x,y) &= 1 - \cos(2\pi x)\cos(2\pi y)\\
p_2(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2}\\
p_3(x,y) &= 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{4}\\
p_4(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{8}\\
p_5(x,y) &= 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{16}\\
p_6(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{32}\\
&\vdots\\
p_n(x,y) &= 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}
\end{aligned}$$

Proof by Induction We show that, by the inductive basis, $p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$ must equal

$$p_1(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(0)}}$$

which a quick check shows is indeed the case.

Next, by the inductive step, we take as true that:

$$p_k(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(k-1)}}$$

Then,

$$\begin{aligned}
p_{k+1}(x,y) &= j\left(\int_0^1 p_1(1-y,z)\cdot p_k(x,y)\,dy\right)\\
&= j\left(\int_0^1 (1 - \cos(2\pi(1-y))\cos(2\pi z))\cdot\left(1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{k-1}}\right)dy\right)
\end{aligned}$$

The product of 1 with itself is 1, and such will integrate to 1 in the unit interval. So we save it. The integrals $\int_0^1 \cos(2\pi y)\,dy$ and $\int_0^1 \cos(2\pi - 2\pi y)\,dy$ both evaluate to zero, so we are left only with the task of evaluating the crossterm:

$$\begin{aligned}\int_0^1 \cos(2\pi(1-y))\cos(2\pi z)\cdot\frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{k-1}}\,dy &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}}\int_0^1 \cos(2\pi-2\pi y)\cos(2\pi y)\,dy\\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}}\int_0^1 \cos^2(2\pi y)\,dy\\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}}\cdot\frac{1}{2}\\ &= -\frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k}}\end{aligned}$$

Let's not forget the 1 we had saved, so:

$$p_{k+1}(x,y) = j\left(1-\frac{\cos(2\pi x)\cos(2\pi z)}{(-2)^k}\right) = 1-\frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^k}$$

as we wanted to show. □

Remark 2.1. ***(April 22, 2010) Probabilistic Interpretation.*** *In direct analogy to the Chapman-Kolmogorov equation, transition from state $y \in [0,1]$ to $x \in [0,1]$ in $n+m$ steps can be achieved by first transitioning to an intermediate state $x^\bullet$ in n steps and then jumping from there to x in m more steps. Particularly, all states are achievable from a starter state since for* Pasquali patch $p(x,y)$ *there are only a finite number of zeroes in the domain because the surface is well-behaved.*

2.2.2. *Other Properties.*

Claim 2.2 (Non-commutativity of *Pasquali Patches* Under the Star Product)**.** ***(October 17, 2010)*** *We know in general the star product is non-commutative (**Claim 1.7**), but we don't know that* Pasquali patches *as a subset of surfaces with domain on* $[0,1]\times[0,1]$ *("*Pasquali patchixes*", due to their resembling matrixes in that they transform functions to other functions, as matrixes transform vectors to other vectors) don't commute.*

Proof by Counterexample Suppose the *Pasquali patches* $p(x,y) = x + \frac{1}{2}$ and $q(x,y) = 1 + xy - \frac{y}{2}$. Then

$$\begin{aligned}p(x,y)\star q(x,y) = p(x)\star q(x,y) &= \int_0^1 p(1-y)\cdot q(x,y)\,dy\\ &= \int_0^1 \left(\frac{3}{2}-y\right)\cdot\left(1+xy-\frac{y}{2}\right)\,dy\\ &= \frac{5x}{12}+\frac{19}{24}\end{aligned}$$

where

$$\begin{aligned}q(x,y)\star p(x,y) &= j\left(\int_0^1 q(1-y,z)\cdot p(x,y)\,dy\right)\\ &= j\left(\int_0^1 q(1-y,z)\cdot p(x)\,dy\right)\\ &= j\left(p(x)\int_0^1 q(1-y,z)\,dy\right)\\ &= j\,(p(x)\cdot u(z)) = p(x)\\ &= x+\frac{1}{2}\end{aligned}$$

□

Corollary 2.3 (Associativity of *Pasquali Patches* Under the Star Product)**.** ***(October 17, 2010, December 3, 2012)*** Pasquali patches *inherit associativity.*

Proof *Pasquali patches* inherit associativity from the star product through **Claim 1.8 Associativity of the Star Product**. □

2.3. Limiting Surface.

Definition 2.2. *The limiting, steady-state, or stationary surface is a surface obtained by taking the limit as n approaches infinity of* Pasquali patch *powers* $\lim_{n\to\infty} p_n(x,y)$*, provided it exists, and we call it* $p_\infty(x,y)$.

Lemma 2.4. ***(October 17, 2010)*** *The stationary surface of*

$$p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$$

is $p_\infty(x,y) = 1$.

Proof Since the *Pasquali patch* power collection of $p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$ can be described by

$$p_n(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}$$

due to **Example 2.2** we need only take the limit as n approaches infinity of this formula:

$$\lim_{n\to\infty} p_n(x,y) = \lim_{n\to\infty}\left(1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}\right) = 1$$

In this particular case, notice that $p_\infty(x,y) = 1$ is also a *Pasquali patch* because

$$\int_0^1 p_\infty(x,y)\,dx = 1$$

□

Lemma 2.5. ***(December 3, 2012)*** *If the sequence $p_n(x,y)$ of* Pasquali patch *powers converge uniformly to $p_\infty(x,y)$, then $p_\infty(x,y)$ is a* Pasquali patch*.*

Proof We want to show that $\int_0^1 p_\infty(x,y)\,dx = 1$. First,

$$\int_0^1 p_\infty(x,y)\,dx = \int_0^1 \lim_{n\to\infty} p_n(x,y)\,dx$$

By uniform convergence of the sequence of *Pasquali patches*, we can exchange the order of the limit to obtain

$$\lim_{n\to\infty}\int_0^1 p_n(x,y)\,dx$$

Now for any n, $p_n(x,y)$ is a Pasquali patch, and therefore the integral is equal to 1 in every case (for any choice of n). Lastly,

$$\lim_{n\to\infty}\{1\}_n = 1$$

and we have arrived at what we wanted to show. □

Claim 2.6. ***(December 3, 2012)*** Pasquali patches *that are functions of y (possibly x) explicitly generate by self-powering a countably infinite collection of* Pasquali patches *that are functions of y (possibly x) explicitly.*

Proof by Induction Let $p(x,y)$ be a *Pasquali patch* that is a function of y (possibly x) explicitly. Thus the basis is true by hypothesis. By the inductive step, let $p_k(x,y)$, $k \in \mathbb{Z}_+$ be an explicit function of y (possibly x). Then:

$$p_{k+1}(x,y) = p_k(x,y) \star p(x,y) = j\left(\int_0^1 p_k(1-y,z)p(x,y)\,dy\right)$$

is an explicit function of y (possibly x) since $p_k(x,y)$ was an explicit function of y (possibly x), and the y was saved by the transformation to z in $p_k(1-y,z)$, so that the integral did not aggregate it. Then, the transformation j takes $z \rightsquigarrow y$ makes $p_{k+1}(x,y)$ an explicit function of y (possibly x). The result is a countable collection of explicit functions of y (possibly x) via self-powering, because $k \in \mathbb{Z}_+$. □

Claim 2.7. ***(October 17, 2010)*** *A* Pasquali patch *star a* Pasquali patch *that is solely a function of x returns the second* Pasquali patch*.*

Proof Suppose the *Pasquali patches* $q(x,y)$ and $p(x,y) = p(x)$. Then:

$$\begin{aligned} q(x,y) \star p(x,y) &= j\left(\int_0^1 q(1-y,z)\cdot p(x)\,dy\right) \\ &= j\left(p(x)\int_0^1 q(1-y,z)\,dy\right) \\ &= j\left(p(x)\cdot u(z)\right) = p(x) \end{aligned}$$

□

Claim 2.8. ***(December 3, 2012)*** Pasquali patches *that are functions of x alone (explicitly or otherwise) generate* Pasquali patches *that are the same as the original* Pasquali patch *via self-powering.*

Proof by Induction $p(x) = p(x)$ by the identity of equality. $p_2(x) = p(x) \star p(x) = p(x)$ by **Claim 2.7**. This establishes the induction basis. Next, suppose $p_k(x) = p(x)$. Then $p_{k+1}(x) = p_k(x) \star p(x) = p(x) \star p(x) = p(x)$, and we are done. □

Claim 2.9. ***(December 3, 2012)*** *If the limiting surface $p_\infty(x, y)$ exists, it is NOT an explicit function of y. It is either an explicit function of x or constant for all x, y. We can describe it WLOG by $p_\infty(x)$.*

Proof The proof consists of two cases.

Case 1. Suppose we have a generator *Pasquali patch* $p(x)$ which is a function of x alone (explicitly or otherwise). Then, by **Claim 2.8**, $\forall n \in \mathbb{Z}_+$, $p_n(x) = p(x)$. Taking the limit as n approaches infinity, we obtain $\lim_{n\to\infty} p_n(x) = \lim_{n\to\infty} p(x)$, or $p_\infty(x) = p(x)$. This limiting surface is therefore also a *Pasquali patch*.

Case 2. In the space of uniformly convergent surfaces generated by *Pasquali patch* self-powers, the limiting surface will exist and will be a *Pasquali patch* itself by **Lemma 2.5**. Now suppose that $p_\infty(x, y)$ is an explicit function of y (possibly x). Then either it belongs to the collection of *Pasquali patch* powers generated by $p(x, y)$, or to a different collection altogether generated by *Pasquali patch* powers of, say, $q(x, y)$. If it belongs to the collection of *Pasquali patch* powers generated by $p(x, y)$, it must be equal to $p_n(x, y)$ for some n. But such is not an accumulation surface because we can generate $p_{n+1}(x, y)$ and onwards. It therefore must belong to a different collection, that generated by $q(x, y)$, and equals $q_m(x, y)$ for some m. The problem is that such isn't an accumulation surface either, being part of a (different) sequence of *Pasquali patch* powers, and in a "parallel" collection. We are faced with a contradiction. $p_\infty(x, y)$ must therefore NOT be an explicit function of y: it is either explicitly or not a function of x, that is, $p_\infty(x)$. Recall it is also a *Pasquali patch* by **Lemma 2.5**.

□

2.4. Probability Distribution Transformations via *Pasquali Patches*.

Claim 2.10. ***(October 10, 2010)*** *A well-behaved continuous probability distribution $b\colon [0, 1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 b(x)\,dx = 1$,* ***star*** *a* Pasquali patch*, yields a continuous probability distribution $c\colon [0, 1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 c(x)\,dx = 1$. In other words, a probability distribution $b(x)$ is taken to a probability distribution $c(x)$ via the* Pasquali patch*: $b(x) \rightsquigarrow c(x)$.*

Proof 1 Let $c(x) = b(x) \star p(x, y)$. We seek to show that

$$\int_0^1 c(x)\,dx = \int_0^1 b(x) \star p(x, y)\,dx = 1$$

Using **Definition 1.1**,

$$\int_0^1 b(x) \star p(x, y)\,dx = \int_0^1 \int_0^1 b(1 - y)p(x, y)\,dy\,dx$$

Absolute convergence of the integrals allows us to exchange the order of integration (Fubini Theorem). Thus:

$$\int_0^1 \int_0^1 b(1 - y)p(x, y)\,dx\,dy = \int_0^1 b(1 - y) \int_0^1 p(x, y)\,dx\,dy$$

The innermost integral adds up to $u(y) = 1$ by **Definition 2.1**. Next

$$\int_0^1 b(1 - y)u(y)\,dy = \int_0^1 b(1 - y) \cdot 1\,dy = 1$$

by virtue of $b(x)$ being a probability distribution. □

Proof 2 Via **Closure of *Pasquali Patches* (Claim 2.1)**, a continuous probability distribution $b(x)$ can be thought of as $b(x, y)$ and a *Pasquali patch*, thus it **star** another *Pasquali patch* will yield a new *Pasquali patch* by closure of *Pasquali patches.* □

Example 2.3. ***(October 10, 2010)*** *Let $b(x) = 6x(1-x)$. This is a beta(2,2) probability distribution and*

$$\int_0^1 (6x(1-x))\,dx = \int_0^1 \left(6x - 6x^2\right)\,dx = \left.(3x^2 - 2x^3)\right|_0^1 = 1$$

is easily checked. Let $p(x,y) = x + \frac{1}{2}$ be a Pasquali patch*, with*

$$\int_0^1 \left(x + \frac{1}{2}\right)\,dx = \left.\left(\frac{x^2}{2} + \frac{x}{2}\right)\right|_0^1 = 1$$

for any choice of y. Then

$$b(x) \star p(x,y) = \int_0^1 b(1-y)p(x,y)\,dy = \int_0^1 6(1-y)(y)\left(x + \frac{1}{2}\right)\,dy = x + \frac{1}{2}$$

This probability distribution has already been shown to integrate to 1 with respect to x.

Corollary 2.11. ***(October 10, 2010)*** *A continuous probability distribution $b\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$* ***star*** *a* Pasquali patch *that solely a function of x yields a continuous probability distribution that is of the same form as the* Pasquali patch*. In other words, a probability distribution $b(x)$ is carried via the* Pasquali patch *$p(x,y) = p(x)$ to $p(x)$: $b(x) \rightsquigarrow p(x)$.*

Proof 1 Using the definition of the star operator, $\int_0^1 b(1-y)p(x,y)\,dy = \int_0^1 b(1-y)p(x)\,dy = p(x)$. □

Proof 2 Via **Claim 2.7**. A continuous probability distribution can be thought of as a *Pasquali patch*, and thus **Claim 2.7** applies. □

2.5. **Fixed Distribution Conjecture.**

Conjecture 2.1. ***(December 3, 2012)*** *It is only natural to ask if there exists a (bounded, well-behaved, probability) function $a(x)$, $a\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ that remains fixed when starred by a* Pasquali patch *$p(x,y)$. In other words, is there an $a(x)$ so that $a(x) \star p(x,y) = a(x)$?*

Claim 2.12. ***(December 3, 2012)*** *Suppose* ***Conjecture 2.1*** *is true. Then $a(x)$ remains fixed for all* Pasquali patch *powers generated by $p(x,y)$.*

Proof by Induction We take as true that $a(x) \star p(x,y) = a(x)$ by **Conjecture 2.1**. By the inductive step, we have that $a(x) \star p_k(x,y) = a(x)$. Then:

$$a(x) \star p_{k+1}(x,y) = a(x) \star (p_k(x,y) \star p(x,y))$$

By **Claim 1.8 Associativity of the Star Product**, this last expression we can write as:

$$(a(x) \star p_k(x,y)) \star p(x,y) = a(x) \star p(x,y) = a(x)$$

and we are done. □

Claim 2.13. ***(December 3, 2012)*** *The only function $b(x)$ that makes true the expression $a(x) \star b(x) = a(x)$, $a(x)$ is a probability distribution $a\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 a(x)\,dx = 1$ (and therefore a* Pasquali patch *from another viewpoint), is $a(x)$ itself.*

Proof 1 We resort to the definition of the star operator, and the LHS is

$$a(x) \star b(x) = \int_0^1 a(1-y)b(x)\,dy = b(x)\int_0^1 a(1-y)\,dy = b(x)$$

On the other hand, the RHS is $a(x)$, and $b(x) = a(x)$. □

Proof 2 By **Claim 2.8**, the LHS is $b(x)$. The RHS is $a(x)$ and the equality is established. □

Corollary 2.14. ***(December 3, 2012)*** $a(x) \star a(x) = a(x)$

Proof By **Claim 2.13**, since the only function of x that makes the expression true is $a(x)$, the result follows by direct plugging-in to the original expression. □

Corollary 2.15. ***(December 3, 2012)*** *$a(x)$ is fixed for the limiting* Pasquali patch *probability distribution of some collection of uniformly convergent* Pasquali patch *self-powers of $p(x,y)$ if and only if $p_\infty(x) = a(x)$.*

Proof We have:

⇒ By **Claim 2.9**, the *Pasquali patch* self-powers of $p(x,y)$ converge to $p_\infty(x)$, a *Pasquali patch* (by **Lemma 2.5**) which is NOT explicitly a function of y. By hypothesis, $a(x) \star p_\infty(x) = a(x)$. Then by **Claim 2.7**, $a(x) \star p_\infty(x) = p_\infty(x)$. It follows that $p_\infty(x) = a(x)$.

⇐ By hypothesis, $p_\infty(x) = a(x)$, and by **Corollary 2.14**, $a(x) \star a(x) = a(x)$. Plugging in the first equation with the second at the appropriate location yields the desired result, and $a(x) \star p_\infty(x) = a(x)$.

□

Corollary 2.16. ***(December 3, 2012)*** *$a(x)$ is fixed for a collection of (uniformly convergent) self-powers of the* Pasquali patch *$p(x,y)$ if and only if it is fixed for the limiting* Pasquali patch *(by **Lemma 2.5**) $p_\infty(x)$.*

Proof We have:

⇒ By hypothesis, $a(x)$ is fixed for all (uniformly convergent) $p_n(x,y)$. Since they are uniformly convergent, they converge to $p_\infty(x)$. Now, $a(x) \star a(x) = a(x)$ is true by **Corollary 2.14**, and $a(x)$ is the ONLY (probability) probability distribution or *Pasquali patch* that makes the statement true by **Claim 2.13**. Also, $p_\infty(x) \star p_\infty(x) = p_\infty(x)$ by **Lemma 2.5** ($p_\infty(x)$ is a *Pasquali patch*) and **Claim 2.8**. It follows that $p_\infty(x) = a(x)$ with these two pieces of information. Lastly, by **Corollary 2.15** (⇐), we have that $a(x) \star p_\infty(x) = a(x)$.

⇐ **(Indirect Proof)** Suppose otherwise, that $a(x) \star p_\infty(x) = a(x)$ and $p_\infty(x) = a(x)$ by **Corollary 2.15**, which in turn implies of **Corollary 2.14** that

$$a(x) \star a(x) = a(x) \Rightarrow p_\infty(x) \star p_\infty(x) = p_\infty(x)$$

but $a(x) \star p(x,y) \neq a(x)$ (for some uniformly convergent $p(x,y)$). Then:

$$(a(x) \star p(x,y)) \star a(x) \neq a(x) \star a(x)$$

$$a(x) \star (p(x,y) \star a(x)) \neq a(x) \star a(x)$$

by **Claim 1.8 Associativity of the Star Product**

$$a(x) \star (a(x)) \neq a(x)$$

by **Claim 2.8** and

$$p_\infty(x) \star p_\infty(x) \neq p_\infty(x)$$

contradicts the last implication of the hypothesis.

□

Remark 2.2 (Guiding Maplet)**.** ***(December 7, 2012)***

The following maplet shows at a high-level the most important claims that are relevant to practically calculating the limiting Pasquali patch*.*

Part I: (graph here)

Part II: (graph here)

Remark 2.3. ***(December 8, 2012)*** *Suppose that, by some sorcery or heuristics or good guess, we have found a candidate probability distribution $a(x)$ so that it is fixed for a given* Pasquali patch *$p(x,y)$. We can hazard the very good guess that the stationary limiting surface is $p_\infty(x) = a(x)$, and this itself is a* Pasquali patch*. It is only a "very good guess" because we have shown this equation to be true for* Pasquali patch *power collections that converge uniformly. In order to be completely certain that $p_\infty(x) = a(x)$, we would have to show that the* Pasquali patch *power collection generated by $p(x,y)$ possesses this property, a task left to, for example, a Weierstrass M-test. The property of uniform convergence becomes unduly burdensome (imagine having to show for each $p(x,y)$ uniform convergence before we can conclude without a shadow of a doubt that the limiting* Pasquali patch *is $p_\infty(x) = a(x)$).*

*We are left with two choices: either (1) we prove for **Construction 2.1** or some-such family of* Pasquali patches *the property of uniform convergence, thus limiting ourselves to the study of these particular* Pasquali patches *(seems overly restrictive from my viewpoint), or (2) we weaken the uniform convergence criterion to just convergence (of any sort). In essence, this second argument implies showing that the limit of the integrals of converging* Pasquali patch *powers generated by $p(x,y)$ equals the integral of the limiting* Pasquali patch*: a revision of **Lemma 2.5** (and subsequent claims that use it). With this, we need only posit convergence and $p_\infty(x) = a(x)$ automatically. This second approach is the one I'm most inclined for and working toward.*

Lemma 2.17. ***(December 3, 2012)*** *If the sequence $p_n(x,y)$ of* Pasquali patch *powers converge ~~uniformly~~ to $p_\infty(x,y)$, then $p_\infty(x,y)$ is a* Pasquali patch*.*

Proof Suppose we have a collection of converging *Pasquali patch* powers generated by the *Pasquali patch* $p(x, y)$. They must converge to something, so call this $p_\infty(x, y)$. Next look at the sequence

$$\left\{ \int_0^1 p_n(x, y)\, dx \right\}_{n \in \mathbb{Z}_+} = \{1\}_n$$

The sequence is obviously bounded above, but in particular, the least upper bound b is 1. Similarly, the infimum is itself 1. Next let's look at $\int_0^1 p_\infty(x, y)\, dx = c$. Suppose that $c > b$. Then there is an integral of some *Pasquali patch* power that must be greater than 1. But this is a contradiction, since all *Pasquali patch* power integrals are 1. Next suppose $c < b$. The issue is that the sequence $\{1\}_n$ is never less than 1 either. Clearly $c = b$ and we are done. □

Things that change with this amendment:

- Proof of **Claim 2.9**. Rather than having Case 2 be the space of ~~uniformly~~ convergent surfaces, it is for convergent surfaces. Also, $p_\infty(x, y)$ is a *Pasquali patch* by **Lemma 2.17**.
- **Corollary 2.15**. Rather than being for ~~uniformly~~ convergent *Pasquali patches*, the corollary holds for convergent *Pasquali patches*: The proof holds with **Lemma 2.17**. This we shall call:

  Corollary 2.18.
- **Corollary 2.16** Again, the corollary holds for convergent *Pasquali patches* and not just ~~uniformly~~ convergent ones. The proof holds with **Lemma 2.17**. This we shall call:

  Corollary 2.19.
- **Guiding Maplet 2.2**. Wherever there is a ~~uniform~~ convergence, we can substitute just "convergence." Wherever there is a **Lemma 2.5**, we can substitute it by **Lemma 2.17**. Wherever there is a **Corollary 2.15**, we can change it to **Corollary 2.18**, and wherever there is a **Corollary 2.16**, we can substitute it to **Corollary 2.19**.

Remark 2.4. ***(December 16, 2012)*** *This is why* ***Example 2.2*** *actually has a limiting probability distribution. The* Pasquali patch *power collection does not converge uniformly (it converges at the nodes first, e.g.), and yet it stabilizes to the uniform* Pasquali patch *eventually.*

A corollary to all this is:

Corollary 2.20. ***(December 16, 2012)*** *$\exists a(x)$ so that $a(x) \star p(x, y) = a(x)$ if and only if the powers of $p(x, y)$ converge.*

Proof We have:

⇒ Suppose not, that the powers $p(x, y)$ do not converge. One way for this to happen and have $\int_0^1 p(x, y)\, dx = 1$ is if the powers get stuck, for example, in a loop, so that, say,

$$p_{2n+1}(x, y) = p(x, y)$$

and

$$p_{2n}(x, y) = q(x, y)$$

and

$$p(x, y) \neq q(x, y)$$

Then by hypothesis $a(x) \star p_{2n+1}(x, y) = a(x)$ and it must also be true that $a(x) \star p_{2n}(x, y) = a(x)$ for this same $a(x)$ (**Claim 2.12**). This implies that $a(x) \star p(x, y) = a(x)$ and $a(x) \star q(x, y) = a(x)$. Now each even powers and odd powers are independently convergent, $p_{2n+1}(x, y)$ to $p(x, y)$ and $p_{2n}(x, y)$ to $q(x, y)$, which implies that $a(x) = p(x, y)$ in the first case and $a(x) = q(x, y)$ in the second case (**Corollary 2.18**). Other than the fact that both $p(x, y)$ and $q(x, y)$ could be functions of x, y, they are defined to be unequal to each other. We have a contradiction.

Next, let's make the loop larger. Suppose that the powers oscillate in k different surface forms, so that we have $p^1(x, y) \ldots p^k(x, y)$ surfaces after which we return to the original. The above would mean that $p_n(x, y) = p^{n \bmod k}(x, y)$. Each "in-between" power is independently convergent to itself after k more powers, and $a(x) = p^k(x, y)$ due to **Corollary 2.18**. But these surfaces were defined to be different from each other, and so, as before, we face a contradiction.

We show of course the $k + 1$ case. Suppose the powers oscillate in $k + 1$ different surface forms, with

$$p^1(x, y) \ldots p^{k+1}(x, y)$$

surfaces after which we return to the original. Then

$$p_n(x,y) = p^{n \bmod (k+1)}(x,y)$$

with each in-between power independently convergent to itself after $k+1$ more powers, and $a(x) = p^{k+1}(x,y)$ due to **Corollary 2.18** each. Each of these surfaces was different. We have a contradiction.

Now we can make the periodicity as large as we like.

This same argument applies for the case in which the oscillatory pattern is established (independently) to m different surfaces eventually. Suppose that we have $p^1(x,y)\dots p^m(x,y)$ surfaces toward which each periodic sequence $p_{n \bmod m}(x,y)$ converges eventually, with

$$\lim_{n\to\infty} p_{n \bmod m}(x,y) = p^{n \bmod m}(x,y)$$

An argument with respect to $a(x)$ and the surfaces of convergence being unequal leads us to contradict, even inductively, that such a scenario is a possibility. An eventual convergent loop cannot form, no matter how large the loop or how slow or fast each convergence.

We have shown essentially that we cannot have an $a(x)$ and have the *Pasquali patch* powers of $p(x,y)$ oscillate in any shape or form in the long run.

We have not shown the "divergent" situation, in which the value of at least one point $p_n(\overline{x},\overline{y}) \to \infty$. In particular, the product $a(1-\overline{y})\cdot p_n(\overline{x},\overline{y}) \to \infty$, and the value of the integral $\int_0^1 a(1-y)\cdot p_n(\overline{x},y)\,dy \to \infty$ as well. Thus $a(\overline{x})$ diverges and $a(x)$ was never bounded or well-behaved (Contradicting **Conjecture 2.1** regarding $a(x)$). NB: This argument can probably be made more rigorous.

Lastly, since the powers neither stabilize oscillatorily nor can they diverge, they must converge.

$\Leftarrow$ **(December 24, 2012)** Since the powers of $p(x,y)$ converge, they converge to $p_\infty(x)$ by **Claim 2.9**, a function of x alone or constant for all x,y. Then it is certainly true that $p(x,y)\star p_\infty(x) = p_\infty(x)$ by **Claim 2.7**. It must also be true by the same claim that $p_\infty(x)\star p_\infty(x) = p_\infty(x)$. If we substitute one equation into the other, we get $p_\infty(x)\star(p(x,y)\star p_\infty(x)) = p_\infty(x)$. By **Claim 1.8, Associativity of the Star Product**, we can rewrite that as $(p_\infty(x)\star p(x,y))\star p_\infty(x) = p_\infty(x)$. But now the arguments in parenthesis must be equal to $p_\infty(x)$, in other words, $p_\infty(x)\star p(x,y) = p_\infty(x)$. Thus $p_\infty(x)$ is fixed for $p(x,y)$. Let $a(x)$ be exactly this $p_\infty(x)$ and we are done.

□

Remark 2.5. ***(December 24, 2012)*** *It is now evident that, so long as we can find a fixed $a(x)$, we know that the powers of $p(x,y)$ (1) converge, (2) the stationary surface is $p_\infty(x)$, that (3) $a(x) = p_\infty(x)$, and (4) $a(x)$ is fixed for all powers, including the stationary surface itself. Conversely, if we know that $p_\infty(x)$ for some $p(x,y)$, then such is in fact $a(x)$. Our efforts from now on should focus on establishing a mechanism to find such $a(x)$.*

2.6. Fixed Distribution Existence.

Claim 2.21. ***(December 2, 2012)*** *For **Construction 2.1**, $a(x)$ exists provided $B = a(x)\star g_1(y)$ converges and can be solved. Its explicit form is*

$$a(x) = p_\infty(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right)B$$

Proof We are looking for $a(x)$ so that $a(x)\star p(x,y) = a(x)$, with

$$p(x,y) = f_1(x)g_1(y) + f_2(x)\left(\frac{1-g_1(y)F_1}{F_2}\right)$$

Using the definition of the star operator (**Definition 1.1**), this is

$$\int_0^1 a(1-y)\left(f_1(x)g_1(y) + f_2(x)\left(\frac{1-g_1(y)F_1}{F_2}\right)\right)dy = a(x)$$

Expansion results in:

$$a(x) = f_1(x)\int_0^1 a(1-y)g_1(y)\,dy + \frac{f_2(x)}{F_2}\cdot 1 - \frac{F_1}{F_2}f_2(x)\int_0^1 a(1-y)g_1(y)\,dy$$

where we have simplified $\int_0^1 a(1-y)\,dy$ to 1 because the transformation to the y-axis does not change the integral result (it remains a probability distribution).

Rearranging, we have

$$\frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right)\int_0^1 a(1-y)g_1(y)\,dy$$

or

$$a(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x) F_1}{F_2} - f_1(x) \right) B = p_\infty(x)$$

(**Remark 2.5**) and derivatives

$$a^i(x) = \frac{f_2^i(x)}{F_2} - \left(\frac{f_2^i(x) F_1}{F_2} - f_1^i(x) \right) B$$

We want to obtain B, but the expression $\int_0^1 a(1-y) g_1(y)\, dy$ has to be clearly defined. We use the tabular method to simplify the integration by parts.

Derivatives	Integrals
$a(1-y)$	$g_1(y)$
$-a'(1-y)$	$G_1^1(y)$
$a''(1-y)$	$G_1^2(y)$
$\vdots$	$\vdots$

Viewed from a different vantage-point, we could have

Derivatives	Integrals
$g_1(y)$	$a(1-y)$
$g_1^{'}(y)$	$-A^1(1-y)$
$g_1^{''}(y)$	$A^2(1-y)$
$\vdots$	$\vdots$

Lastly, we have:

$$B = a(1-y) G_1^1(y) + a^{'}(1-y) G_1^2(y) + \ldots \Big|_0^1 = \sum_{i=0}^{\infty} a^i(1-y) G_1^{i+1}(y) \Big|_0^1$$

or

$$B = -g_1(y) A^1(1-y) - g_1^{'}(y) A^2(1-y) - \ldots \Big|_0^1 = -\sum_{i=0}^{\infty} g_1^i(y) A^{i+1}(1-y) \Big|_0^1$$

If the sum diverges we are stuck, but if the sum converges we are good. □

Corollary 2.22. ***(December 25, 2012)*** Pasquali patches *constructed as by* ***Construction 2.1*** *with (finite) polynomial function choices for $f_1(x)$ and $f_2(x)$ are guaranteed to have a fixed $a(x) = p_\infty(x)$ regardless of (integrable) function choice $g_1(y)$. Similarly,* ***Construction 2.1*** Pasquali patches *with a (finite) polynomial choice for $g_1(y)$ are guaranteed to have such fixed $a(x)$ as well, regardless of (integrable) function choices for $f_1(x)$ and $f_2(x)$.*

Proof Since

$$B = a(1-y) G_1^1(y) + a^{'}(1-y) G_1^2(y) + \ldots \Big|_0^1 = \sum_{i=0}^{\infty} a^i(1-y) G_1^{i+1}(y) \Big|_0^1$$

and the derivatives of $a(x)$ are eventually zero (for all subsequent derivatives), the sum itself is finite. Thus, B converges, which implies that **Claim 2.21** applies.

Next, since

$$B = -g_1(y) A^1(1-y) - g_1^{'}(y) A^2(1-y) - \ldots \Big|_0^1 = -\sum_{i=0}^{\infty} g_1^i(y) A^{i+1}(1-y) \Big|_0^1$$

and the derivatives of $g_1(y)$ are eventually zero (including all subsequent derivatives), such sum is also finite and B converges. Again, **Claim 2.21** applies. □

Example 2.4. ***(January 16, 2011)*** *In* ***Example 2.1****, we had the* Pasquali patch

$$p(x, y) = x^2 y^3 + x \left(2 - \frac{2y^3}{3} \right)$$

with $f_1(x) = x^2$, $f_2(x) = 2x$, $g_1(y) = y^3$. This implies by ***Claim 2.21*** *that*

$$a(x) = p_\infty(x) = 2x - \left(\frac{2x}{3} - x^2 \right) B$$

with derivatives

$$a'(x) = 2 - \left(\frac{2}{3} - 2x\right) B$$

$$a''(x) = -2B$$

Specifically,

$$\begin{array}{rcr} a(1) = 2 + \frac{B}{3} & \text{and} & a(0) = 0 \\ a'(1) = 2 + \frac{4B}{3} & \text{and} & a'(0) = 2 - \frac{2B}{3} \\ a''(1) = -2B & \text{and} & a''(0) = -2B \end{array}$$

Next, we want to calculate B:

$$B = \cancel{a(0)G_1^1(1)} + a'(0)G_1^2(1) + a''(0)G_1^3(1) - \cancel{\left(a(1)G_1^1(0) + a'(1)G_1^2(0) + a''(1)G_1^3(0)\right)}$$

The parenthetical part dies because integrals of y^3 *evaluated at* 0 *vanish. So does the first term since* $a(0) = 0$. *So we are left with:*

$$B = a'(0)G_1^2(1) + a''(0)G_1^3(1) = \frac{6-2B}{3} \cdot \frac{1}{20} + -2B \cdot \frac{1}{120}$$

which solves

$$B = \frac{6}{63} = \frac{2}{21}$$

Thus, we have that

$$a(x) = p_\infty(x) = 2x - \left(\frac{2x}{3} - x^2\right)\frac{2}{21} = \frac{2x^2}{21} + \frac{122x}{63}$$

For a consistency check, by ***Lemma 2.17*** *this is a* Pasquali patch *and therefore*

$$\int_0^1 \left(\frac{2x^2}{21} + \frac{122x}{63}\right) dx = 1$$

can be verified to indeed be the case.

Remark 2.6. *Practically speaking, all functions* $f(x), g(y)$ *with Taylor polynomial representations in the domain* $[0,1]$ *will converge (just truncate them at the appropriate precision, e.g.).*

2.7. **Probability Distribution Transformations via *Pasquali Patch* Power Collections.**

Claim 2.23. ***(January 12, 2013)*** *Take a* Pasquali patch *power collection union its limiting surface,* $\mathbb{P} \cup p_\infty(x) = \mathbb{P}^\infty$*, and a well-behaved, bounded probability distribution* $c(x)$*, with* $c\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$*, and* $\int_0^1 c(x)\,dx = 1$*. Then* $c(x) \star \mathbb{P}^\infty = \mathbb{C}$ *is a collection of transformed probability distributions of x (explicit or not).*

Proof by Induction The proof consists of two cases.

Case 1. The generator of $\mathbb{P}^\infty$ is a *Pasquali patch* function of x alone (explicitly or constant). Then, by **Claim 2.8** all *Pasquali patches* in the collection are functions of x (explicitly or constant), and in fact all powers in $\mathbb{P}$ equal $p(x)$, including $p_\infty(x)$ (**Case 1 of Claim 2.9**). Begin with

$$c(x) \star P^1 = c(x) \star p(x) = p(x)$$

by **Claim 2.7**. Assume that $c(x) \star P^k = p(x)$. Then

$$c(x) \star P^{k+1} = c(x) \star (P^1 \star P^k) = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star p(x) = p(x) \star p(x) = p(x)$$

We used **Claim 1.9 Powering Symmetry** to exchange the order of the powers, **Claim 1.8 Associativity of the Star Product**, and again by **Claim 2.7** the last equality holds. Furthermore, since $p_\infty(x) = p(x)$, $c(x) \star p_\infty(x) = c(x) \star p(x) = p(x)$. This last part is justified by **Claim 2.7** once more. Thus, $c(x) \star \mathbb{P}^\infty = \mathbb{C} = \{p(x)\}$ and all functions are explicit functions of x or constant as we wanted to show. By **Claim 2.10**, the collection is made up of probability distributions on $[0,1]$.

Case 2. The generator of $\mathbb{P}^\infty$ is a *Pasquali patch* function of y explicitly. Begin by

$$c(x) \star P^1 = \int_0^1 c(1-y)p(x,y)\,dy = c_1(x)$$

Since we are integrating with respect to y we can see that this function $c_1(x)$ is either explicitly function of x or constant. Next assume that $c(x) \star P^k = c_k(x)$ is an explicit function of x or constant. Then

$$c(x) \star P^{k+1} = c(x) \star P^{1+k} = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star P^1 = c_k(x) \star P^1$$

This last part is $\int_0^1 c_k(1-y)p(x,y)\,dy$, which, by virtue of integrating in terms of y yields an explicit function of x or constant. Now take $c(x) \star p_\infty(x) = p_\infty(x)$ by **Claims 2.9** and **2.7**. Thus, all functions in the collection $c(x) \star \mathbb{P}^\infty = \mathbb{C}$ are explicit functions of x or constant. Finally, by **Claim 2.10**, the collection is made up of probability distributions on $[0,1]$.

□

Claim 2.24. ***(January 12, 2013)*** *Let $c(x) \star P^m = c_m(x)$, the mth transform of $c(x)$ via the mth power of $p(x,y)$. An equivalent way to obtain the mth transform is by $c_{m-1}(x) \star P^1 = c_m(x)$.*

Proof by Induction Begin by $c(x) \star P^1 = c_1(x)$ using the first statement and $c_0(x) \star P^1 = c_k(x)$ using the second statement. Clearly the two statements are equivalent. Next, assume that the statement holds for the kth transform, so that $c(x) \star P^k = c_k(x)$ and $c_{k-1}(x) \star P^1 = c_k(x)$. Now $c(x) \star P^{k+1} = c_{k+1}(x)$ by definition. An equivalent way to write this is

$$c(x) \star P^{1+k} = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star P^1 = c_k(x) \star P^1$$

Here we have used **Claim 1.8 Associativity of the Star Product**. Thus $c_k(x) \star P^1 = c_{k+1}(x)$, and we are done. □

Claim 2.25. ***(January 13, 2013)*** *The collection $\mathbb{C}$ converges to $p_\infty(x)$.*

Indirect Proof 1 Suppose $\lim_{m\to\infty} c_m(x) \neq p_\infty(x)$. Then

$$\left[\lim_{m\to\infty} c_m(x)\right] \star p_\infty(x) \neq p_\infty(x) \star p_\infty(x)$$

and

$$p_\infty(x) \neq p_\infty(x) \star p_\infty(x)$$

is a contradiction of **Claim 2.8** or **Claim 2.7**. Thus $\lim_{m\to\infty} c_m(x) = p_\infty(x)$. □

Indirect Proof 2 Suppose that the sequence of probability distributions $c_m(x)$ diverge, that

$$\lim_{m\to\infty} c_m(\overline{x}) = \infty$$

for some $\overline{x}$. This would have to mean that $\lim_{m\to\infty} \int_0^1 c(1-y)p_m(\overline{x},y)\,dy = \infty$. But then the sequence $p_m(\overline{x},y)$ would have to be divergent, because $c(x)$ is chosen to be well-behaved and bounded. We have a contradiction, because the sequence $p_m(x,y)$ is well-behaved and bounded by virtue of being *Pasquali patch* powers with bounded first-power generator and fixed volume. Thus $\lim_{m\to\infty} c_m(x)$ converges.

Next suppose the sequence of probability distributions $c_m(x)$ does not converge to $p_\infty(x)$ but to some other probability distribution in the collection $\mathbb{C}$, say $c_k(x)$. The issue is this isn't an accumulation probability distribution, since we can generate $c_{k+1}(x)$ by starring by P^1 on the right (**Claim 2.24**). Lastly, say we choose a probability distribution outside of the collection $\mathbb{C}$, in some other collection $\mathbb{D}$. If we choose $d_k(x)$, this isn't an accumulation probability distribution for $\mathbb{C}$ because it was generated by *Pasquali patch* $q(x,y)$, and also $d_{k+1}(x)$ can be generated by right-starring by Q^1. Then pick $p^d_\infty(x)$, the limiting probability distribution in $\mathbb{D}$. There's just no way to tie such to the generator collection $\mathbb{P}^\infty$, since it is an accumulation probability distribution for $\mathbb{D}$, not for $\mathbb{C}$. The only choice left (that makes sense) is $p^c_\infty(x)$. □

Corollary 2.26. ***(January 13, 2013)***

$$\lim_{m\to\infty} c_m(x) = p_\infty(x)$$

if and only if

$$\lim_{m\to\infty} [c(x) \star P^m] = c(x) \star \left[\lim_{m\to\infty} P^m\right]$$

In other words, we can pull the limiting process under the integral.

Proof We have:

⇒ By definition, $c(x) \star P^m = c_m(x)$. Then

$$\lim_{m\to\infty} c_m(x) = \lim_{m\to\infty} [c(x) \star P^m] = p_\infty(x)$$

by hypothesis.

Next,

$$\lim_{m\to\infty} [c(x) \star P^m] = \lim_{m\to\infty} \int_0^1 c(1-y) p_m(x,y)\, dy = p_\infty(x)$$

An equivalent expression can be found through **Claim 2.7**:

$$p_\infty(x) = \int_0^1 c(1-y) p_\infty(x)\, dy$$

This is,

$$p_\infty(x) = \int_0^1 c(1-y) \left[\lim_{m\to\infty} p_m(x,y)\right] dy$$

In other words, we now have

$$p_\infty(x) = c(x) \star \left[\lim_{m\to\infty} P^m\right]$$

Putting these two expressions together yields the desired result.

⇐ Take

$$c(x) \star \left[\lim_{m\to\infty} P^m\right] = c(x) \star p_\infty(x) = p_\infty(x)$$

by definition of the limiting surface and **Claim 2.7**. The alternative way to write this by the stated equality in the hypothesis is:

$$\lim_{m\to\infty} [c(x) \star P^m] = p_\infty(x)$$

In turn, this can be written as $\lim_{m\to\infty} c_m(x) = p_\infty(x)$ by definition and we are done.

□

Corollary 2.27. ***(January 15, 2013)***

$$\lim_{m\to\infty} \int_0^1 p_m(x,y)\, dy = p_\infty(x) = \lim_{m\to\infty} p_m(x,y)$$

Proof Take $c(x) = u(x)$, the uniform probability distribution, which equals 1 for all x in the domain. We can then state by **Corollary 2.26** that

$$\lim_{m\to\infty} [1 \star P^m] = 1 \star \left[\lim_{m\to\infty} P^m\right]$$

which is equivalent to:

$$\lim_{m\to\infty} \int_0^1 p_m(x,y)\, dy = \int_0^1 \lim_{m\to\infty} p_m(x,y)\, dy$$

which, in turn, yields

$$\lim_{m\to\infty} \int_0^1 p_m(x,y)\, dy = p_\infty(x) = \lim_{m\to\infty} p_m(x,y)$$

□

Remark 2.7. ***(January 15, 2013)*** *We now have that*

$$\lim_{m\to\infty} \int_0^1 p_m(x,y)\, dx = 1$$

by ***Definition 2.1*** *and* ***Claim 2.1 Closure of*** **Pasquali Patches** *and*

$$\lim_{m\to\infty} \int_0^1 p_m(x,y)\, dy = p_\infty(x)$$

by ***Corollary 2.27****.*

Corollary 2.28. ***(January 13, 2013)***

$$\lim_{m\to\infty} c_m(x) = p_\infty(x)$$

if and only if

$$\lim_{m\to\infty} \left[c_{m-1}(x) \star P^1\right] = \left[\lim_{m\to\infty} c_{m-1}(x)\right] \star P^1$$

Proof This follows from **Claim 2.24**. □

Corollary 2.29. ***(January 13, 2013)***

$$\lim_{m\to\infty} c_m(x) = p_\infty(x)$$

if and only if

$$c(x) \star \left[\lim_{m\to\infty} P^m\right] = \left[\lim_{m\to\infty} c_{m-1}(x)\right] \star P^1$$

In other words,

$$c(x) \star p_\infty(x) = c_\infty(x) \star P^1$$

Proof This follows from **Corollary 2.26** and **Corollary 2.28**. □

3. Dynamics

Corollary 3.1. ***(April 6, 2013)*** *If*

$$\lim_{t\to\infty} p_t(x,y) = p_\infty(x)$$

then

$$\lim_{t\to\infty} \Delta P^t = 0$$

Proof Take

$$\Delta P^t = P^{t+1} - P^t$$

Applying the limit at infinity we get

$$\begin{aligned}\lim_{t\to\infty} \Delta P^t &= \lim_{t\to\infty} \left[P^{t+1} - P^t\right] \\ &= \lim_{t\to\infty} P^{t+1} - \lim_{t\to\infty} P^t \\ &= P^\infty - P^\infty \\ &= 0\end{aligned}$$

where the next-to-last step is justified by the hypothesis of the corollary (in other words, we assume convergence to $p_\infty(x)$). □

Corollary 3.2. ***(March 31, 2013)*** *If*

$$\lim_{t\to\infty} c_t(x) = p_\infty(x)$$

then

$$\lim_{t\to\infty} \Delta c_t(x) = 0$$

Proof 1 From **Claim 2.24**,

$$c_{t+1}(x) = c_0(x) \star P^{t+1}$$

Thus we have that

$$\begin{aligned}c_{t+1}(x) - c_t(x) &= c_0(x) \star P^t - c_0(x) \star P^{t-1} \\ &= c_0(x) \star \left(P^t - P^{t-1}\right)\end{aligned}$$

where this part is justified by **Corollary 1.5, Distributive Property of the Star Operator**. Next at steady-state

$$\begin{aligned}\lim_{t\to\infty} \Delta c_t(x) &= \lim_{t\to\infty} \left[c_0(x) \star \left(P^t - P^{t-1}\right)\right] \\ &= c_0(x) \star \left(\left[\lim_{t\to\infty} P^t\right] - \left[\lim_{t\to\infty} P^{t-1}\right]\right) \\ &= c_0(x) \star (P^\infty - P^\infty) \\ &= c_0(x) \star 0 \\ &= 0\end{aligned}$$

Where we have pulled the limit under the star operator by using the hypothesis and conclusion of **Corollary 2.26**, **Corollary 3.1**, and we also used **Claim 1.6 Zero Property of the Star Product**. □

Proof 2 Using the **Claim 2.24** equivalence, we have

$$c_{t+1}(x) = c_t(x) \star P^1$$

and

$$c_{t+1}(x) - c_t(x) = (c_t(x) - c_{t-1}(x)) \star P^1$$

Taking the limit as $t \to \infty$ we get:

$$\begin{aligned}\lim_{t\to\infty} \Delta c_t(x) &= \lim_{t\to\infty} (c_t(x) - c_{t-1}(x)) \star P^1 \\ &= \lim_{t\to\infty} \left(c_t(x) \star P^1 - c_{t-1}(x) \star P^1\right) \\ &= \lim_{t\to\infty} \left[c_t(x) \star P^1\right] - \lim_{t\to\infty} \left[c_{t-1} \star P^1\right] \\ &= \left[\lim_{t\to\infty} c_t(x)\right] \star P^1 - \left[\lim_{t\to\infty} c_{t-1}(x)\right] \star P^1 \\ &= c_\infty(x) \star P^1 - c_\infty(x) \star P^1 \\ &= (c_\infty(x) - c_\infty(x)) \star P^1 \\ &= 0 \star P^1 \\ &= 0\end{aligned}$$

in which we used the hypothesis and limit implication of **Corollary 2.28**. Here we also used **Claim 1.6 Zero Property of the Star Product**. □

******************************begin in-progress******************************

Construction 3.1. ***(March 31, 2013)*** *Let $k(x,t)$ is a piecewise continuous linear* functional *so that at each unit interval the slope is $\Delta c_t(x)$ for any x, and at every t it is equal to $c_t(x)$ star-weighted by an arbitrary $g(y), g\colon [0,1] \to \mathbb{R}$, in other words, $c_t(x) \star g(y)$.*

Claim 3.3. ***(March 31, 2013)*** *If*

$$\lim_{t\to\infty} c_t(x) = p_\infty(x)$$

then

$$\lim_{t\to\infty} \frac{\partial k(x,t)}{\partial t} = 0$$

Proof For the intervals between $c_{t+1}(x)$ and $c_t(x)$, the derivative is $\Delta c_t(x)$ by construction. Thus we can take

$$\lim_{t\to\infty} \frac{\partial k(x,t)}{\partial t} = \lim_{t\to\infty} \Delta c_t(x) = 0$$

by **Claim 3.2**. Note that at infinity, convergence of $c_t(x)$ eventually smooths any kinks at integer time-steps and allows for the derivative to be defined there. □

Remark 3.1. ***(March 31, 2013)*** *The above claim intends to construct a situation in which probability on a* Pasquali patch *will accumulate or un-accumulate uniformly (linearly) in each unit of time interval, but stabilizing in the long-run,* for each *x*.

Remark 3.2. ***(March 31, 2013)*** *Anatomically, it seems clear that, because of its recursive dependence, $c_t(x)$ has basic form:*

$$c_t(x) = \overbrace{I(x)}^{\text{invariant in time}} + W(x) \cdot \overbrace{c_{t-1}(x) \star g(y)}^{\text{variable in time}}$$

The idea is to create a "continuous continuation" of $c_t(x)$ by using $k(x,t)$.

Claim 3.4. ***(March 31, 2013)*** *Take*

$$c_t^\circ(x) = \overbrace{I(x)}^{\text{invariant in time}} + W(x) \cdot \overbrace{k(x,t-1)}^{\text{variable in time}}$$

with the usual supposition of convergence to $p_\infty(x)$ of $c_t(x)$. Then the partial derivative

$$\lim_{t\to\infty} \frac{\partial c_t^\circ(x)}{\partial t} = 0$$

Proof By taking the partial derivative in the intervals where it is defined (except at $t \in \mathbb{Z}^+$ initially, but then for practical purposes the convergence smooths the derivative at these points too), we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{\partial}{\partial t}\left[I(x) + W(x)k(x,t-1)\right] &= W(x) \lim_{t\to\infty} \frac{\partial}{\partial t}\left[k(x,t-1)\right] \\ &= W(x)\cdot 0 \\ &= 0\end{aligned}$$

where we used **Claim 3.3**. □

Remark 3.3. *The newly created function $c_t^{\circ}(x)$ is now continuous in t and not just defined at integer time-steps. This in essence describes a manner in which to define in-between powers of* Pasquali patches*. Sometimes it is possible to find $k(x,t)$ explicitly, as the next example shows.*

*******************************end in-progress*****************************

4. Relevant Generalizations

4.1. Surface Trace or *str*.

Definition 4.1. ***(March 3, 2013)*** *Take the function $h\colon [0,1]^2 \to \mathbb{R}$. The* surface trace *is the value of the integral of the diagonal $y = -x+1$ or $x = 1-y$ of such surface. In other words, it is:*

$$\operatorname{str}\left[h(x,y)\right] = \int_0^1 h(x,1-x)\,dx = \int_0^1 h(1-y,y)\,dy$$

4.2. Specific Finite-Dot-Product Surfaces.

Definition 4.2. ***(March 3, 2013)*** *A specific finite-dot-product surface is a function $h\colon [0,1]^2 \to \mathbb{R}$ such that:*

$$h(x,y) = \sum_{k=1}^{n} f_k(x) g_k(y) = \mathbf{f}(\mathrm{x})\cdot \mathbf{g}(\mathrm{y})$$

4.2.1. *Specific Finite-Dot-Product Surface Trace (*str*).*

Claim 4.1. ***(March 3, 2013)*** *The specific finite-dot-product surface trace is:*

$$\operatorname{str}\left[h(x,y)\right] = \sum_{k=1}^{n} C_{k,k} = \operatorname{tr}\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n}\end{bmatrix}$$

where $C_{k,k} = f_k(x) \star g_k(x)$.

Proof by Induction We have:

$$\operatorname{str}\left[h(x,y)\right] = \int_0^1 h(x,1-x)\,dx = \int_0^1 h(1-y,y)\,dy$$

by **Definition 4.1**. We can think of the specific finite-dot-product surface

$$h(x,y) = f_1(x) g_1(y)$$

and readily calculate the trace str in this way:

$$\begin{aligned}\operatorname{str}\left[h(x,y)\right] = \int_0^1 h(1-y,y)\,dy &= \int_0^1 f_1(1-y) g_1(y)\,dy \\ &= \int_0^1 f_1(1-y) g_1(y)\,dy \\ &= f_1(x) \star g_1(y) \\ &= C_{1,1} \\ &= \operatorname{tr}\left[C_{1,1}\right]\end{aligned}$$

This constitutes the base case. So assume that the mth case is true, and

$$\begin{aligned}
\operatorname{str}[h(x,y)] = \sum_{k=1}^{m} C_{k,k} &= \operatorname{tr}\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,m} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m,1} & C_{m,2} & \cdots & C_{m,m} \end{bmatrix} \\
&= \sum_{k=1}^{m}\left(\int_0^1 f_k(1-y)g_k(y)\,dy\right) \\
&= \int_0^1 \left(\sum_{k=1}^{m} f_k(1-y)g_k(y)\right) dy
\end{aligned}$$

We now show that the $m+1$ case works as well. So let

$$h(x,y) = \sum_{k=1}^{m+1} f_k(x)g_k(y) = \sum_{k=1}^{m} f_k(x)g_k(y) + f_{m+1}(x)g_{m+1}(y)$$

Then the surface trace can be calculated as

$$\begin{aligned}
\operatorname{str}[h(x,y)] = \int_0^1 h(1-y,y)\,dy &= \int_0^1 \left(\sum_{k=1}^{m} f_k(1-y)g_k(y) + f_{m+1}(1-y)g_{m+1}(y)\right) dy \\
&= \int_0^1 \left(\sum_{k=1}^{m} f_k(1-y)g_k(y)\right) dy + \int_0^1 f_{m+1}(1-y)g_{m+1}(y)\,dy \\
&= \sum_{k=1}^{m} C_{k,k} + C_{m+1,m+1} \\
&= \sum_{k=1}^{m+1} C_{k,k} \\
&= \operatorname{tr}\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,m} & C_{1,m+1} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,m} & C_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{m,1} & C_{m,2} & \cdots & C_{m,m} & C_{m,m+1} \\ C_{m+1,1} & C_{m+1,2} & \cdots & C_{m+1,m} & C_{m+1,m+1} \end{bmatrix}
\end{aligned}$$

□

4.2.2. *Specific Finite-Dot-Product Surface Eigenvalues.*

Claim 4.2. ***(March 25, 2011)*** *Let $e\colon [0,1] \to \mathbb{R}$ be smooth and well-behaved, and $h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y)$ with $h\colon [0,1]^2 \to \mathbb{R}$ likewise. Then there are* ***two*** *λ values that satisfy*

$$e(x) \star h(x,y) = \lambda\, e(x)$$

provided

$$\begin{aligned}
C_{1,1} &= f_1(x) \star g_1(y) \\
C_{1,2} &= f_1(x) \star g_2(y) \\
C_{2,1} &= f_2(x) \star g_1(y) \\
C_{2,2} &= f_2(x) \star g_2(y)
\end{aligned}$$

converge.

Proof In **Claim 4.2**, the equation

$$e(x) \star h(x,y) = \lambda\, e(x)$$

can be written out specifically as

$$\int_0^1 e(1-y)h(x,y)\,dy = \lambda\, e(x)$$

More explicitly, this is:

$$\begin{aligned}\lambda\, e(x) &= \int_0^1 e(1-y)\,(f_1(x)g_1(y) + f_2(x)g_2(y))\,dy \\ &= f_1(x)\int_0^1 e(1-y)g_1(y)\,dy + f_2(x)\int_0^1 e(1-y)g_2(y)\,dy \\ &= B_1 f_1(x) + B_2 f_2(x)\end{aligned}$$

where B_1, B_2 are constants. If we divide by λ as

$$e(x) = \frac{B_1}{\lambda} f_1(x) + \frac{B_2}{\lambda} f_2(x)$$

then the equation must hold provided $\lambda \neq 0$. So we have excluded an eigenvalue right from the start.

We can systematically write the derivatives of $e(x)$:

$$\begin{array}{ccc} e(x) & = & \frac{B_1}{\lambda} f_1(x) + \frac{B_2}{\lambda} f_2(x) \\ e'(x) & = & \frac{B_1}{\lambda} f_1'(x) + \frac{B_2}{\lambda} f_2'(x) \\ e''(x) & = & \frac{B_1}{\lambda} f_1''(x) + \frac{B_2}{\lambda} f_2''(x) \\ \vdots & \vdots & \vdots \\ e^k(x) & = & \frac{B_1}{\lambda} f_1^k(x) + \frac{B_2}{\lambda} f_2^k(x) \\ \vdots & \vdots & \vdots \end{array}$$

again with $\lambda \neq 0$. We want to calculate the constants B_1, B_2, to see if they are restricted in some way by a formula, and we do this by integrating by parts as we did before. Thus, we have that if

$$B_1 = \int_0^1 e(1-y)g_1(y)\,dy$$

the tabular method gives:

Derivatives	Integrals
$e(1-y)$	$g_1(y)$
$-e'(1-y)$	$G_1^1(y)$
$e''(1-y)$	$G_1^2(y)$
$\vdots$	$\vdots$

and so,

$$\begin{aligned}B_1 &= \int_0^1 e(1-y)g_1(y)\,dy \\ &= e(1-y)G_1^1(y)\big|_0^1 + e'(1-y)G_1^2(y)\big|_0^1 + \dots \\ &= \sum_{i=0}^{\infty} e^i(1-y)G_1^{i+1}(y)\Bigg|_0^1\end{aligned}$$

if we remember the alternating sign of the multiplications, and we are allowed some leeway in notation. Ultimately, this last bit means: $\sum_{i=0}^{\infty} e^i(0)G_1^{i+1}(1) - \sum_{i=0}^{\infty} e^i(1)G_1^{i+1}(0)$. Since we have already explicitly written the derivatives of $e(x)$, the $e^i(0), e^i(1)$ derivatives can be written as

$$\frac{B_1}{\lambda} f_1^i(0) + \frac{B_2}{\lambda} f_2^i(0)$$

and

$$\frac{B_1}{\lambda} f_1^i(1) + \frac{B_2}{\lambda} f_2^i(1)$$

respectively. We have then:

$$B_1 = \sum_{i=0}^{\infty}\left(\frac{B_1}{\lambda} f_1^i(0) + \frac{B_2}{\lambda} f_2^i(0)\right) G_1^{i+1}(1) - \sum_{i=0}^{\infty}\left(\frac{B_1}{\lambda} f_1^i(1) + \frac{B_2}{\lambda} f_2^i(1)\right) G_1^{i+1}(0)$$

Since we aim to solve for B_1, multiplying by λ makes things easier, and also we must rearrange all elements with B_1 in them, so we get:

$$\lambda B_1 = B_1 \sum_{i=0}^{\infty} \left(f_1^i(0) G_1^{i+1}(1) - f_1^i(1) G_1^{i+1}(0) \right) + B_2 \sum_{i=0}^{\infty} \left(f_2^i(0) G_1^{i+1}(1) - f_2^i(1) G_1^{i+1}(0) \right)$$

Subtracting both sides the common term and factoring the constant we endeavor to solve for, we get:

$$\left(\lambda - \sum_{i=0}^{\infty} \left(f_1^i(0) G_1^{i+1}(1) - f_1(1) G_1^{i+1}(0) \right) \right) B_1 = B_2 \sum_{i=0}^{\infty} \left(f_2^i(0) G_1^{i+1}(1) - f_2^i(1) G_1^{i+1}(0) \right)$$

or

$$B_1 = \frac{B_2 \sum_{i=0}^{\infty} f_2^i(1-y) G_1^{i+1}(y) \Big|_0^1}{\lambda - \sum_{i=0}^{\infty} f_1^i(1-y) G_1^{i+1}(y) \Big|_0^1} = \frac{B_2 \left(f_2(x) \star g_1(y) \right)}{\lambda - \left(f_1(x) \star g_1(y) \right)} = \frac{B_2 C_{2,1}}{\lambda - C_{1,1}}$$

A similar argument for B_2 suggests

$$B_2 = \frac{B_1 \sum_{i=0}^{\infty} f_1^i(1-y) G_2^{i+1}(y) \Big|_0^1}{\lambda - \sum_{i=0}^{\infty} f_2^i(1-y) G_2^{i+1}(y) \Big|_0^1} = \frac{B_1 \left(f_1(x) \star g_2(y) \right)}{\lambda - \left(f_2(x) \star g_2(y) \right)} = \frac{B_1 C_{1,2}}{\lambda - C_{2,2}}$$

where the new constants introduced emphasizes the expectation that the sums (or integrals) converge. Plugging in the one into the other we get:

$$B_1 = \frac{\left(\frac{B_1 C_{1,2}}{\lambda - C_{2,2}} \right) C_{2,1}}{\lambda - C_{1,1}} = \frac{B_1 C_{1,2} C_{2,1}}{(\lambda - C_{2,2})(\lambda - C_{1,1})}$$

and now we have additional restrictions on lambda: $\lambda \neq C_{2,2}$ and $\lambda \neq C_{1,1}$. Furthermore, the constant B_1 drops out of the equation, suggesting these constants can be anything we can imagine (all of $\mathbb{R}$ without restriction), but then we have the constraint:

$$(\lambda - C_{2,2})(\lambda - C_{1,1}) = C_{1,2} C_{2,1}$$

(Notice how this equation can be put in determinant form!

$$\det \begin{vmatrix} C_{1,1} - \lambda & C_{1,2} \\ C_{2,1} & C_{2,2} - \lambda \end{vmatrix} = 0$$

This form of the equation becomes the basis of **Claim 4.4**.)

Expanding the equation suggests:

$$\lambda^2 - (C_{2,2} + C_{1,1})\lambda + (C_{1,1} C_{2,2} - C_{1,2} C_{2,1}) = 0$$

which we can solve by the quadratic equation of course, as:

$$\lambda_{1,2} = \frac{(C_{2,2} + C_{1,1}) \pm \sqrt{(C_{2,2} - C_{1,1})^2 + 4 C_{1,2} C_{2,1}}}{2}$$

So not only is λ not equal to many values, it is incredibly restricted to two of them.

Now the constants $C_{1,1}, C_{1,2}, C_{2,1}, C_{2,2}$ have been expressed in terms of the integration-by-parts sums in the expectation that subsequent derivatives of $f_{1,2}(x)$ will eventually vanish (or are periodically 0 [1]). There is nothing to stop us from redefining them in terms of $g_{1,2}(y)$ derivatives instead, if these were to vanish quicker or were to force the sum convergence where the derivatives of $f_{1,2}(x)$ did not.[2]

Now, figuring out the eigenfunctions $e(x)$ that go together with these eigenvalues is an exercise in finding constants B_1 and B_2 for a given eigenvalue. [3] □

*************************begin in-progress*******************************

[1]Need to clarify

[2]For this reason it may be convenient to leave the integration-by-parts method open-ended by rewriting and redefining shorthand

$$f_1(x) \star g_1(y) = C_{1,1}$$
$$f_2(x) \star g_1(y) = C_{2,1}$$
$$f_1(x) \star g_2(y) = C_{1,2}$$
$$f_2(x) \star g_2(y) = C_{2,2}$$

like we did.

[3]Need to show

Claim 4.3. ***(January 26, 2013)*** Pasquali patches *constructed as by* ***Construction 2.1*** *always have a* $\lambda = 1$ *eigenvalue.*

[In progress] □

*************************end in-progress******************************

Claim 4.4. ***(January 26, 2013)*** *The allowable values of* λ *is equal to the number of (pair) function terms*

$$h(x,y) = \sum_{k=1}^{n} f_k(x) g_k(y)$$

has, provided pairwise star products between functions of x *and functions of* y *converge. In particular, these can be found by*

$$\det|A - \lambda I| = 0$$

where A *is an* $n \times n$ *matrix of pairwise star products* $f_i(x) \star g_j(y)$ *(we call such constants* $C_{i,j}$*) with* $i, j \in \{1 \ldots n\}$*, and* I *is the identity matrix. This creates a characteristic equation, in direct analogy to how we obtain eigenvalues in linear algebra contexts.*

Proof The base case has already been shown in **Claim 4.2**. Suppose that

$$h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y) + \ldots + f_k(x)g_k(y)$$

and as before, we are looking for

$$e(x) \star h(x,y) = \lambda\, e(x)$$

Arguing similarly as in **Claim 4.2**, we end up with the linear system:

$$\begin{aligned}
\lambda B_1 &= B_1 C_{1,1} + B_2 C_{2,1} + \ldots + B_k C_{k,1} \\
\lambda B_2 &= B_1 C_{1,2} + B_2 C_{2,2} + \ldots + B_k C_{k,2} \\
&\vdots \\
\lambda B_k &= B_1 C_{1,k} + B_2 C_{2,k} + \ldots + B_k C_{k,k}
\end{aligned}$$

for which we can write the augmented matrix

$$\left[\begin{array}{cccc|c}
C_{1,1} - \lambda & C_{2,1} & \ldots & C_{k,1} & 0 \\
C_{1,2} & C_{2,2} - \lambda & \ldots & C_{k,2} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{1,k} & C_{2,k} & \ldots & C_{k,k} - \lambda & 0
\end{array}\right]$$

Now the determinant of the square matrix must be equal to 0, otherwise there is exactly one solution for constants $B_1, B_2, \ldots, B_k$ (we are specifically looking that these constants be *any* value, so the matrix must be singular and consequently the determinant equal to zero). Thus we have that

$$\det \begin{vmatrix}
C_{1,1} - \lambda & C_{2,1} & \ldots & C_{k,1} \\
C_{1,2} & C_{2,2} - \lambda & \ldots & C_{k,2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1,k} & C_{2,k} & \ldots & C_{k,k} - \lambda
\end{vmatrix} = 0$$

The $k+1$th case can be argued similarly. With

$$h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y) + \ldots + f_k(x)g_k(y) + f_{k+1}(x)g_{k+1}(y)$$

we get:

$$\begin{aligned}
\lambda B_1 &= B_1 C_{1,1} + B_2 C_{2,1} + \ldots + B_k C_{k,1} \\
\lambda B_2 &= B_1 C_{1,2} + B_2 C_{2,2} + \ldots + B_k C_{k,2} \\
&\vdots \\
\lambda B_k &= B_1 C_{1,k} + B_2 C_{2,k} + \ldots + B_k C_{k,k} \\
\lambda B_{k+1} &= B_1 C_{1,k+1} + B_2 C_{2,k+1} + \ldots + B_{k+1} C_{k+1,k+1}
\end{aligned}$$

and so we endeavor to solve the augmented matrix:

$$\left[\begin{array}{ccccc|c} C_{1,1}-\lambda & C_{2,1} & \dots & C_{k,1} & C_{k+1,1} & 0 \\ C_{1,2} & C_{2,2}-\lambda & \dots & C_{k,2} & C_{k+1,2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C_{1,k} & C_{2,k} & \dots & C_{k,k}-\lambda & C_{k+1,k} & 0 \\ C_{1,k+1} & C_{2,k+1} & \dots & C_{k,k+1} & C_{k+1,k+1}-\lambda & 0 \end{array}\right]$$

so that the vector $B_1, B_2, \dots, B_k, B_{k+1}$ admits an infinity of solutions. We are thus again looking for the singular matrix with determinant zero:

$$\det \begin{vmatrix} C_{1,1}-\lambda & C_{2,1} & \dots & C_{k,1} & C_{k+1,1} \\ C_{1,2} & C_{2,2}-\lambda & \dots & C_{k,2} & C_{k+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,k} & C_{2,k} & \dots & C_{k,k}-\lambda & C_{k+1,k} \\ C_{1,k+1} & C_{2,k+1} & \dots & C_{k,k+1} & C_{k+1,k+1}-\lambda \end{vmatrix} = 0$$

which we can now simplify to

$$\det |A - \lambda I| = 0$$

as we wanted. □

*************************begin in-progress*******************************

Conjecture 4.1. *If λ is an eigenvalue of the surface $h(x, y)$, with $h\colon [0,1]^2 \to \mathbb{R}$, it is an eigenvalue of all the powers of $h(x, y)$.*

[In progress] □

4.2.3. *Functions on Eigenvalues of Specific Finite-Dot-Product Surfaces.*
sdet.

Definition 4.3 (Specific Finite-Dot-Product Surface Determinant or *sdet*)**.** ***(February 10, 2013)*** *Specifically in a linear-algebra context, the determinant of a square matrix is the product of its eigenvalues. When we talk about the specific finite-dot-product surface*

$$h(x, y) = \sum_{k=1}^{n} f_k(x) g_k(y)$$

we shall ***define*** *the* surface determinant *or* sdet *as the product of the eigenvalues it generates:*

$$\operatorname{sdet} |h(x, y)| = \prod_{k=1}^{n} \lambda_k$$

Claim 4.5. ***(February 10, 2013)*** *The* sdet *for*

$$h(x, y) = f_1(x) g_1(y) + f_2(x) g_2(y)$$

is

$$\operatorname{sdet} |h(x, y)| = C_{1,1} C_{2,2} - C_{1,2} C_{2,1} = \det \begin{vmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{vmatrix}$$

Proof This follows directly from the eigenvalue formulation:

$$\lambda_{1,2} = \frac{(C_{2,2} + C_{1,1}) \pm \sqrt{(C_{2,2} - C_{1,1})^2 + 4 C_{1,2} C_{2,1}}}{2}$$

The product is:

$$\begin{aligned}\lambda_1 \cdot \lambda_2 &= \left(\frac{(C_{2,2}+C_{1,1}) + \sqrt{(C_{2,2}-C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}\right) \cdot \left(\frac{(C_{2,2}+C_{1,1}) - \sqrt{(C_{2,2}-C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}\right) \\ &= \frac{(C_{2,2}+C_{1,1})^2 - (C_{2,2}-C_{1,1})^2 - 4C_{1,2}C_{2,1}}{4} \\ &= \frac{4C_{1,1}C_{2,2} - 4C_{1,2}C_{2,1}}{4} \\ &= C_{1,1}C_{2,2} - C_{1,2}C_{2,1} \\ &= \det \begin{vmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{vmatrix}\end{aligned}$$

□

Conjecture 4.2. ***(February 23, 2013)*** *The* sdet *of a specific finite surface with n pair functions of x and y terms is:*

$$\text{sdet}\,|h(x,y)| = \det \begin{vmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \end{vmatrix}$$

str (Revisited).

Claim 4.6. ***(March 3, 2013)*** *In the context of specific finite-dot-product surfaces,*

$$\text{str}\,[h(x,y)] = \sum_{k=1}^{n} \lambda_k$$

Proof by Induction From the eigenvalue formulation of **Claim 4.2**, we have

$$\lambda_{1,2} = \frac{(C_{2,2}+C_{1,1}) \pm \sqrt{(C_{2,2}-C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}$$

The sum is:

$$\begin{aligned}\lambda_1 + \lambda_2 &= \left(\frac{(C_{2,2}+C_{1,1}) + \sqrt{(C_{2,2}-C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}\right) + \left(\frac{(C_{2,2}+C_{1,1}) - \sqrt{(C_{2,2}-C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}\right) \\ &= C_{1,1} + C_{2,2}\end{aligned}$$

which is exactly $\text{str}\,[f_1(x)g_1(y) + f_2(x)g_2(y)]$. This constitutes the base case. Next assume that it is true that

$$\text{str}\left[\sum_{k=1}^{m} f_k(x)g_k(y)\right] = \sum_{k=1}^{m} C_{k,k} = \sum_{k=1}^{m} \lambda_k$$

Then

$$\begin{aligned}\text{str}\left[\sum_{k=1}^{m+1} f_k(x)g_k(x)\right] &= \sum_{k=1}^{m-1} C_{k,k} + C_{m,m} + C_{m+1,m+1} \\ &=\end{aligned}$$

□

*************************end in-progress*******************************

4.3. Specific Infinite-Dot-Product Surfaces.

Remark 4.1. ***(February 23, 2013)*** *There is no reason why we should restrict ourselves to the study of specific finite-dot-product surfaces, where we could make the leap to specific infinite-dot-product surfaces of the form*

$$h(x,y) = \sum_{k=0}^{\infty} f_k(x)g_k(y)$$

Conjecture 4.3. ***(February 23, 2013) Conjecture ??*** *suggests that the* str *of a specific infinite-dot-product surface is:*

$$\operatorname{str}\left[h(x,y)\right] = \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} C_{k,k}$$

provided all $C_{k,k}$ converge in their calculation and the sum itself converges.

4.4. **Specific Finite-Dot-Product Surface Specific Infinite-Dot-Product Representations.**

Example 4.1. ***(February 23, 2013)*** *Recall the Maclaurin series expansion*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We can think of this equation as

$$\overbrace{e^x}^{f_1^\circ(x)} \cdot \overbrace{1}^{g_1^\circ(y)} = \overbrace{1}^{f_1(x)} \cdot \overbrace{1}^{g_1(y)} + \overbrace{x}^{f_2(x)} \cdot \overbrace{1}^{g_2(y)} + \overbrace{\frac{x^2}{2!}}^{f_3(x)} \cdot \overbrace{1}^{g_3(y)} + \overbrace{\frac{x^3}{3!}}^{f_4(x)} \cdot \overbrace{1}^{g_4(y)} + \cdots$$

In other words, a specific finite-dot-product surface on the LHS and a specific infinite-dot-product surface on the RHS. The LHS has sum of eigenvalues (eigenvalue)

$$e^x \star 1 = \int_0^1 e^{(1-y)}\, dy = e - 1$$

by the trivial case of ***Claim 4.4****, and the RHS is calculated directly by*

$$\begin{aligned}
&= \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(1-y)^k}{k!} \right) dy \\
&= \sum_{k=0}^{\infty} \left(\int_0^1 \frac{(1-y)^k}{k!}\, dy \right) \\
&= \sum_{k=0}^{\infty} \left(- \frac{(1-y)^{(k+1)}}{(k+1)k!} \bigg|_0^1 \right) \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!}
\end{aligned}$$

where we have pulled the integral inside the sum due to absolute convergence and the infinite sum of factorial reciprocals therefore of course also converges.[4] *This gives credence to* ***Conjecture 4.3****, since the second row is exactly the* str *of a specific infinite-dot-product surface*

$$\operatorname{str}\left| \sum_{k=0}^{\infty} \frac{x^k}{k!} \right| = 1 \star 1 + x \star 1 + \frac{x^2}{2!} \star 1 + \frac{x^3}{3!} \star 1 + \cdots = \sum_{k=1}^{\infty} \lambda_k$$

Claim 4.7. ***(February 23, 2013)*** *Take the Taylor-expandable function $f(x)$, $f\colon [0,1]\times[0,1] \to \mathbb{R}$*

$$f(x) = \sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i$$

[4]One easily checks that the LHS and the RHS are indeed equivalent

$$e - 1 = \sum_{k=1}^{\infty} \frac{1}{k!}$$

by noticing that the Maclaurin expansion of e^x evaluated at $x = 1$ yields

$$e = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}$$

with $a \in [0,1]$. *Then its eigenvalue is the convergent series sum*

$$\lambda = \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right)$$

with $a \in [0,1]$.

Proof We have that

$$\overbrace{f(x)}^{f_1^\circ(x)} \cdot \overbrace{1}^{g_1^\circ(y)} = \overbrace{\left(\sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i \right)}^{f_1^\circ(x)} \cdot \overbrace{1}^{g_1^\circ(y)}$$

So then the LHS eigenvalue can be calculated by the trivial case of **Claim 4.4**, and the RHS can be calculated by direct application of star operator

$$\lambda = \left(\sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i \right) \star 1$$

If we believe **Conjecture 4.3**, this RHS can be interpreted as the sum-of-eigenvalues of a specific infinite-dot-product surface.

We therefore have

$$\lambda = \int_0^1 \left(\sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(1-y-a)^i \right) dy$$

by definition. By absolute convergence of the sum, we can bring in the integral and solve:

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} \left(\int_0^1 (1-y-a)^i \, dy \right) \\
&= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} \left(\frac{-(1-y-a)^{i+1}}{i+1} \bigg|_0^1 \right) \\
&= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{i+1} \right) \\
&= \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right)
\end{aligned}$$

with $a \in [0,1]$. □

Corollary 4.8. ***(February 23, 2013)*** *The Maclaurin-expandable* $f(x), f\colon [0,1] \times [0,1] \to \mathbb{R}$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^i(0)}{i!} x^i$$

has eigenvalue

$$\lambda = \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!}$$

Proof We have that by **Claim 4.7**, the Taylor-expandable $f(x)$ has eigenvalue

$$\lambda = \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right)$$

with $a \in [0,1]$. Letting $a = 0$, we get

$$\begin{aligned}
\lambda &= \sum_{i=0}^{\infty} f^i(0) \cdot \left(\frac{(1)^{i+1} - (0)^{i+1}}{(i+1)!} \right) \\
&= \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!}
\end{aligned}$$

□

Corollary 4.9. ***(February 25, 2013)*** *The Taylor-expansion of $f(x), f\colon [0,1]\times[0,1]\to\mathbb{R}$ about $a=1$ has eigenvalue*

$$\lambda = \sum_{i=0}^{\infty} \frac{(-1)^i f^i(1)}{(i+1)!} = \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!}$$

Proof This follows directly from **Claim 4.7**, with $f(x)$ has eigenvalue

$$\lambda = \sum_{i=0}^{\infty} f^i(a)\cdot\left(\frac{(1-a)^{i+1}-(-a)^{i+1}}{(i+1)!}\right)$$

At $a=1$, this becomes

$$\begin{aligned}\lambda &= \sum_{i=0}^{\infty} f^i(1)\cdot\left(\frac{(0)^{i+1}-(-1)^{i+1}}{(i+1)!}\right)\\ &= \sum_{i=0}^{\infty} \frac{(-1)^i f^i(1)}{(i+1)!}\\ &= \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!}\end{aligned}$$

□

Claim 4.10. ***(February 25, 2013)*** *For any (infinitely-differentiable, Taylor-expandable) function $f(x), f\colon [0,1]\times[0,1]\to\mathbb{R}$,*

$$\sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} = \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!}$$

Proof Since both **Corollary 4.8** and **Corollary 4.9** are equal to the specific finite-surface eigenvalue, we have the following:

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{f^i(0)}{(i+1)!} &= \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!}\\ \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(0)}{(i+1)!} + \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(0)}{(i+1)!} &= \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!}\\ \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} &= \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!}\end{aligned}$$

□

Remark 4.2. ***(February 25, 2013)*** *It's rather neat that **Claim 4.10** relates the even and odd derivatives of functions, essentially stating that they are constrained in a very specific way at two particular points.*

Example 4.2. ***(February 26, 2013)*** *Take for example the rather unassuming function $f(x)=(x+1)^2, f\colon [0,1]^2\to\mathbb{R}$. We have:*

$$\begin{array}{ccc} f(x)=(x+1)^2 & f(1)=4 & f(0)=1\\ f'(x)=2(x+1) & f'(1)=4 & f'(0)=2\\ f''(x)=2 & f''(1)=2 & f''(0)=2\\ f'''(x)=0 & f'''(1)=0 & f'''(0)=0\\ \vdots & \vdots & \vdots\end{array}$$

*Let's calculate the odd part as in the LHS of **Claim 4.10**:*

$$\sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} = \frac{4+2}{2!} = 3$$

The even part is:

$$\sum_{\substack{i\ even \\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!} = \frac{4-1}{1!}+\frac{2-2}{3!}=3$$

And the eigenvalue that gave rise to this invariant is:

$$\lambda=\sum_{i=0}^{\infty}\frac{f^i(0)}{(i+1)!}=\frac{1}{1!}+\frac{2}{2!}+\frac{2}{3!}=1+1+\frac{1}{3}=7/3$$

Claim 4.11. ***(April 14, 2013)*** *Let*

$$s^o(n)=\sum_{i=1}^{n}\frac{1}{(2i)!}$$

and

$$s^e(n)=\sum_{i=0}^{n}\frac{1}{(2i+1)!}$$

Then

$$\lim_{n\to\infty}\frac{s^e(n)}{s^o(n)}=\frac{e+1}{e-1}\approx 2.164$$

In other words, the infinite sum $\lim_{n\to\infty} s^e(n)$ *is approximately* 116% *larger than the infinite sum* $\lim_{n\to\infty} s^o(n)$.

Proof Another rather interesting example analogous to **Example 4.2** is the function $f(x)=e^x$, with $f\colon [0,1]^2\to\mathbb{R}$. Here since $f^i(x)=e^x, \forall i\in\mathbb{Z}^+\cup\{0\}$, we readily evaluate $f^i(1)=e$ and $f^i(0)=1$ for all non-negative i. Thus we get:

$$\sum_{\substack{i\ \text{odd} \\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} = \frac{e+1}{2!}+\frac{e+1}{4!}+\dots$$

$$=(e+1)\cdot\left(\frac{1}{2!}+\frac{1}{4!}+\dots\right)$$

and

$$\sum_{\substack{i\ \text{even} \\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!} = \frac{e-1}{1!}+\frac{e-1}{3!}+\dots$$

$$=(e-1)\cdot\left(\frac{1}{1!}+\frac{1}{3!}+\dots\right)$$

Since these two equations must be equal by **Claim 4.10**, we get

$$\frac{e+1}{e-1}=\frac{\left(\frac{1}{1!}+\frac{1}{3!}+\dots\right)}{\left(\frac{1}{2!}+\frac{1}{4!}+\dots\right)}=\frac{\sum_{i=0}^{\infty}\frac{1}{(2i+1)!}}{\sum_{i=1}^{\infty}\frac{1}{(2i)!}}=\frac{\lim_{n\to\infty}s^e(n)}{\lim_{n\to\infty}s^o(n)}=\lim_{n\to\infty}\frac{s^e(n)}{s^o(n)}$$

□

Claim 4.12. ***(April 14, 2013)*** *The eigenvalue for the function* $f(x)=e^x$*, with* $f\colon[0,1]^2\to\mathbb{R}$*, is*

$$\lambda_{e^x}=e-1$$

Proof Since, by **Claim 4.8**, we have

$$\lambda_{e^x}=\sum_{i=0}^{\infty}\frac{f^i(0)}{(i+1)!}$$

and $f^i(0)=1$ for all non-negative i, the substitution yields:

$$\lambda=\sum_{i=0}^{\infty}\frac{1}{(i+1)!}=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

Next, notice that the Maclaurin expansion of e^x:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

evaluated at $x = 1$ yeids

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Thus, we get that

$$e - 1 = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \lambda_{e^x}$$

and we are done. □

Claim 4.13. ***(April 14, 2013)*** *Another expression for λ_{e^x} is*

$$\lambda_{e^x} = \lim_{n\to\infty} (s^o(n) + s^e(n)) = \lim_{n\to\infty} s^o(n) + \lim_{n\to\infty} s^e(n)$$

Proof Since

$$\lambda_{e^x} = e - 1 = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

and

$$\lim_{n\to\infty} s^o(n) = \sum_{i=1}^{\infty} \frac{1}{(2i)!} = \frac{1}{2!} + \frac{1}{4!} + \cdots$$

and

$$\lim_{n\to\infty} s^e(n) = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

we can easily see that

$$\lim_{n\to\infty} s^e(n) + \lim_{n\to\infty} s^o(n) = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \lambda_{e^x}$$

as we wanted to show. Thus we have that

$$\lim_{n\to\infty} s^e(n) + \lim_{n\to\infty} s^o(n) = e - 1$$

□

Corollary 4.14. ***(April 14, 2013)*** *The infinite sum*

$$\lim_{n\to\infty} s^o(n) = \sum_{i=1}^{\infty} \frac{1}{(2i)!} = \frac{(e-1)^2}{2e}$$

On the other hand the infinite sum

$$\lim_{n\to\infty} s^e(n) = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} = \frac{e^2-1}{2e}$$

Thus, both infinite sums are convergent.

Proof 1 We have that

$$\lim_{n\to\infty} s^o(n) + \lim_{n\to\infty} s^e(n) = e - 1$$

by **Claim 4.13**. Next, using the ratio

$$\frac{\lim_{n\to\infty} s^e(n)}{\lim_{n\to\infty} s^o(n)} = \frac{e+1}{e-1}$$

from **Claim 4.11**, we get that

$$\lim_{n\to\infty} s^e(n) = \frac{e+1}{e-1} \cdot \lim_{n\to\infty} s^o(n)$$

Thus it must be true that

$$\lim_{n\to\infty} s^o(n) + \frac{e+1}{e-1} \cdot \lim_{n\to\infty} s^o(n) = \left(1 + \frac{e+1}{e-1}\right) \lim_{n\to\infty} s^o(n) = e - 1$$

which, through direct algebraic manipulation yields:

$$\lim_{n\to\infty} s^o(n) = \frac{e-1}{1 + \frac{e+1}{e-1}} = \frac{(e-1)\cdot(e-1)}{\left(1 + \frac{e+1}{e-1}\right)\cdot(e-1)} = \frac{(e-1)^2}{2e}$$

Next,

$$\lim_{n\to\infty} s^e(n) = e - 1 - \lim_{n\to\infty} s^o(n) = e - 1 - \frac{(e-1)^2}{2e}$$
$$= \frac{2e^2 - 2e - \left(e^2 - 2e + 1\right)}{2e}$$
$$= \frac{e^2 - 1}{2e}$$

□

Proof 2 We can use the Maclaurin expansions of

$$\sinh(x) = \sum_{i=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

and

$$\cosh(x) = \sum_{i=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

evaluated at $x = 1$. Therefore we have

$$\cosh(1) = \sum_{i=0}^{\infty} \frac{1}{(2i)!} = \frac{e^2+1}{2e}$$

Thus

$$\sum_{i=1}^{\infty} \frac{1}{(2i)!} = \frac{e^2+1}{2e} - 1 = \frac{e^2+1}{2e} - \frac{2e}{2e} = \frac{(e-1)^2}{2e}$$

On the other hand we have:

$$\sinh(1) = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} = \frac{e^2-1}{2e}$$

□

Claim 4.15. ***(February 26, 2013)*** *For any (infinitely-differentiable, Taylor-expandable) function $f(x), f\colon [0,1] \times [0,1] \to \mathbb{R}$,*

$$\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(1) + f^i(0)}{(i+1)!} = \int_0^1 \left(\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(x)}{i!} \right) dx$$

Proof An alternate form of **Claim 4.10** follows from shifting the sum indices:

$$\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(1) + f^i(0)}{(i+1)!} = \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^{i+1}(1) - f^{i+1}(0)}{i!}$$

The RHS is, in essence:

$$= \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{\int_0^1 f^i(x)\,dx}{i!}$$
$$= \int_0^1 \left(\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(x)}{i!} \right) dx$$

where this last step follows from the fact that sums of subsequences must converge (the odd subsequence), thus allowing us to bring out the integral. □

Claim 4.16. ***(March 14, 2013)*** *The finite polynomial function $f\colon [0,1]^2 \to \mathbb{R}$*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

has eigenvalue

$$\lambda = \sum_{i=0}^{n} \frac{a_i}{i+1}$$

Proof by Induction Take

$$f(x) = a_0$$

with a_0 is a constant. The eigenvalue is

$$\lambda = f(x) \star 1 = \int_0^1 a_0 \, dy = a_0$$

which, using the formula, is

$$\lambda = \sum_{i=0}^{0} \frac{a_i}{i+1} = \frac{a_0}{1}$$

and the equivalence is established for the base case. Let's assume the formula works in the kth case. We show the $k+1$th case. So take

$$f(x) = a_{k+1}x^{k+1} + a_k x^k + \ldots + a_1 x + a_0$$

Which has eigenvalue

$$\begin{aligned}
\lambda = f(x) \star 1 &= \int_0^1 \Big(a_{k+1}(1-y)^{k+1} + a_k(1-y)^k + \ldots + a_1(1-y) + a_0 \Big) \, dy \\
&= \int_0^1 a_{k+1}(1-y)^{k+1} \, dy + \sum_{i=0}^{k} \frac{a_i}{i+1} \\
&= \left. \frac{-a_{k+1}(1-y)^{k+2}}{k+2} \right|_0^1 + \sum_{i=0}^{k} \frac{a_i}{i+1} \\
&= \frac{a_{k+1}}{k+2} + \sum_{i=0}^{k} \frac{a_i}{i+1} \\
&= \sum_{i=0}^{k+1} \frac{a_i}{i+1}
\end{aligned}$$

□

Corollary 4.17. ***(March 14, 2013)*** *For finite polynomial functions,*

$$\sum_{i=0}^{n} \frac{f^i(0)}{(i+1)!} = \sum_{i=0}^{n} \frac{a_i}{i+1} = \sum_{i=0}^{n} \frac{(-1)^i f^i(1)}{(i+1)!}$$

Proof This is a consequence of **Claim 4.8**, **Claim 4.9**, and **Claim 4.16**.

Remark 4.3. ***Corollary 4.17*** *essentially relates the constant (last) derivative of each term of a finite polynomial function with the coefficients of such function, and also the (sum of) coefficients of each derivative to the coefficients of the original function.*

Claim 4.18. ***(March 14, 2013)*** *The eigenvalue for the function* $f\colon [0,1]^2 \to \mathbb{R}$, $f(x) = a\sin(bx+c)$ *for constants* a, b, c *is*

$$\lambda_{f(x)} = \frac{a}{b}\left(\cos(c) - \cos(b+c)\right)$$

In particular, if $b \in \{(2m+1)\pi\}_{m\in\mathbb{Z}}$ *and* $c \in \{2\pi n\}_{n\in\mathbb{Z}}$

$$\lambda_{f(x)} = \frac{2a}{(2m+1)\pi}$$

The eigenvalue for the function $g\colon [0,1]^2 \to \mathbb{R}$, $g(x) = a\cos(bx+c)$ *is*

$$\lambda_{g(x)} = \frac{a}{b}\left(\sin(b+c) - \sin(c)\right)$$

If $b, c \in \{\pi n\}_{n\in\mathbb{Z}}$*, then*

$$\lambda_{g(x)} = 0$$

Proof For $f(x)$, the eigenvalue is

$$\begin{aligned}\lambda_{f(x)} = f(x) \star 1 = a \int_0^1 \sin(b(1-y)+c)\, dy \\ = a \left(\frac{\cos(b(1-y)+c)}{b} \right) \Big|_0^1 \\ = \frac{a}{b} \left(\cos(c) - \cos(b+c) \right)\end{aligned}$$

Choosing $b \in \{(2m+1)\pi\}_{m\in\mathbb{Z}}$ and $c \in \{2\pi n\}_{n\in\mathbb{Z}}$ yields

$$\lambda_{f(x)} = \frac{a}{(2m+1)\pi} (1+1) = \frac{2a}{(2m+1)\pi}$$

Next, take $g(x)$ with eigenvalue

$$\begin{aligned}\lambda_{g(x)} = g(x) \star 1 = a \int_0^1 \cos(b(1-y)+c)\, dy \\ = a \left(\frac{\sin(b(1-y)+c)}{-b} \right) \Big|_0^1 \\ = \frac{a}{b} \left(\sin(b+c) - \sin(c) \right)\end{aligned}$$

It is easy to see that with $b, c \in \{\pi n\}_{n\in\mathbb{Z}}$ we have

$$\lambda_{g(x)} = \frac{a}{b}(0+0) = 0$$

Corollary 4.19. ***(March 16, 2013)***

$$\sum_{\substack{i\ even \\ 0\le i\le\infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!} = \sum_{\substack{i\ odd \\ 0\le i\le\infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!}$$

with $n \in \mathbb{Z}$.

Proof Using the Maclaurin expansion of $\cos(\pi n x + \pi n)$, $n \in \mathbb{Z}$ (notice $a = 1, b = \pi n, c = \pi n$), we get:

$$\cos(\pi n x + \pi n) = \sum_{i=0}^{\infty} \frac{(-1)^i (\pi n x + \pi n)^{2i}}{(2i)!}$$

and eigenvalue, by **Claim 4.18**,

$$\begin{aligned}
0 &= \int_0^1 \left(\sum_{i=0}^{\infty} \frac{(-1)^i(\pi n(1-y)+\pi n)^{2i}}{(2i)!} \right) dy \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \int_0^1 (\pi n(1-y)+\pi n)^{2i}\, dy \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \left(-\frac{(\pi n(1-y)+\pi n)^{2i+1}}{\pi n(2i+1)} \right)\Bigg|_0^1 \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \left(\frac{-(\pi n)^{2i+1}+(2\pi n)^{2i+1}}{\pi n(2i+1)} \right) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i(\pi n)^{2i}}{(2i+1)!} \\
0 &= \sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{(\pi n)^{2i}}{(2i+1)!} - \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{(\pi n)^{2i}}{(2i+1)!} \\
\sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{(\pi n)^{2i}}{(2i+1)!} &= \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{(\pi n)^{2i}}{(2i+1)!} \\
\sum_{\substack{i \text{ even} \\ 0\le i\le\infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!} &= \sum_{\substack{i \text{ odd} \\ 0\le i\le\infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!}
\end{aligned}$$

□

Claim 4.20. ***(April 6, 2013)*** *The Bernoulli polynomials*

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i$$

with $B_n\colon [0,1]^2 \to \mathbb{R}$ *and* b_m *are the Bernoulli numbers, have eigenvalue*

$$\lambda_{B_n(x)} = \sum_{i=0}^{n} \binom{n}{i} \frac{b_{n-i}}{i+1}$$

Proof Begin with

$$\begin{aligned}
\lambda_{B_n(x)} &= B_n(x) \star 1 \\
&= \int_0^1 B_n(1-y)\, dy \\
&= \int_0^1 \sum_{i=1}^{n} \binom{n}{i} b_{n-i}(1-y)^i\, dy \\
&= \sum_{i=1}^{n} \binom{n}{i} b_{n-i} \int_0^1 (1-y)^i\, dy \\
&= \sum_{i=1}^{n} \binom{n}{i} b_{n-i}\ \frac{-(1-y)^{i+1}}{i+1}\Bigg|_0^1 \\
&= \sum_{i=0}^{n} \binom{n}{i} \frac{b_{n-i}}{i+1}
\end{aligned}$$

□

Lemma 4.21. ***(April 6, 2013)*** *The kth derivative of the Bernoulli polynomials,* $B_n\colon [0,1]^2 \to \mathbb{R}$*, is*

$$\frac{\mathrm{d}^k B_n(x)}{\mathrm{d}x^k} = \sum_{i=k}^{n} \binom{n}{i} b_{n-i} ({}_iP_k) x^{i-k}$$

Proof by Induction [[In Progress]] □

Lemma 4.22. ***(April 6, 2013)*** *An alternative form for the kth derivative of the Bernoulli polynomials, $B_n\colon [0,1]^2 \to \mathbb{R}$, is*

$$\frac{\mathrm{d}^k B_n(x)}{\mathrm{d}x^k} = \binom{n}{k} b_{n-k} k! + \sum_{i=k+1}^{n} \binom{n}{i} b_{n-i} ({}_iP_k) x^{i-k}$$

Proof by Induction The zeroth derivative is

$$\frac{\mathrm{d}^0 B_n(x)}{\mathrm{d}x^0} = \binom{n}{0} b_n + \sum_{i=1}^{n} \binom{n}{i} b_{n-i} ({}_iP_0) x^{i} = \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i$$

is exactly the explicit function of Bernoulli polynomials. The first derivative is

$$\frac{\mathrm{d} B_n(x)}{\mathrm{d}x} = \binom{n}{1} b_{n-1} + \sum_{i=2}^{n} \binom{n}{i} b_{n-i} ({}_iP_1) x^{i-1} = \sum_{i=1}^{n} \binom{n}{i} b_{n-i} i x^{i-1}$$

So now we assume the kth term works. The $k+1$th term is:

$$\frac{\mathrm{d}^{k+1} B_n(x)}{\mathrm{d}x^{k+1}} = \binom{n}{k+1} b_{n-(k+1)} (k+1)! + \sum_{i=k+2}^{n} \binom{n}{i} b_{n-i} ({}_iP_{k+1}) x^{i-(k+1)} = \sum_{i=k+1}^{n} \binom{n}{i} b_{n-i} ({}_iP_{k+1}) x^{i-(k+1)}$$

□

Corollary 4.23. ***(April 6, 2013)*** *For Bernoulli polynomials, $B_n\colon [0,1]^2 \to \mathbb{R}$,*

$$\left.\frac{\mathrm{d^k} B_n(x)}{\mathrm{d}x^k}\right|_{x=0} = \binom{n}{k} b_{n-k} k!$$

Proof This follows directly from evaluating the formula of **Lemma 4.22** at $x = 0$. □

Claim 4.24. ***(April 6, 2013)*** *An alternative formulation of Bernoulli polynomial eigenvalues is*

$$\lambda_{B_n(x)} =$$

Proof Using **Corollary 4.8**, we have

$$\lambda = \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!}$$

and for Bernoulli polynomials

$$f^i(0) = \binom{n}{i} b_{n-i} i!$$

by **Corollary 4.23**. Thus we have: □

*****************begin in progress*****************

Claim 4.25. ***(April 6, 2013)***

$$\lambda_{B_n(x)} = 0$$

for $n \in \mathbb{Z}^+$ with $b_1 = -\frac{1}{2}$. Notice the explicit exclusion when $n = 0$.

Proof by Induction First note that if $n = 0$ we get

$$\lambda_{B_0(x)} = \sum_{i=0}^{0} \binom{0}{i} \frac{b_0}{i+1} = \binom{0}{0} \frac{b_0}{1} = 1$$

(recall that $b_0 = 1$) so we must discard it. Next look at the base case $n = 1$

$$\begin{aligned}
\lambda_{B_1(x)} &= \sum_{i=0}^{1} \binom{1}{i} \frac{b_{n-1}}{i+1} \\
&= \binom{1}{0} \frac{b_1}{1} + \binom{1}{1} \frac{b_0}{2} \\
&= -\frac{1}{2} + \frac{1}{2} \\
&= 0
\end{aligned}$$

Assume that the kth Bernoulli polynomial eigenvalue is zero too, so that

$$\sum_{i=0}^{k} \binom{k}{i} \frac{b_{k-i}}{i+1} = 0$$

and we show the $k+1$th case:

$$\lambda_{B_{k+1}(x)} = \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{b_{k+1-i}}{i+1}$$

Using Pascal's identity[5] we get

$$\begin{aligned}\lambda_{B_{k+1}(x)} &= \sum_{i=0}^{k+1} \left(\binom{k}{i} + \binom{k}{i-1} \right) \frac{b_{k+1-i}}{i+1} \\ &= \sum_{i=0}^{k} \left(\binom{k}{i} + \binom{k}{i-1} \right) \frac{b_{k-i}}{i+1}\end{aligned}$$

*****************end in progress*********************

5. Equivalencies of the Star Operator

6. Other Claims and Proofs

6.1. **Combinatorics.** The following identity arose in my studies of the Mexican lottery.

Claim 6.1 (Elisa and Carlos Pasquali Combinatorial Identity)**.** ***(December 27, 2008)***

$$\binom{n-s}{r-s} \cdot \binom{n}{s} = \binom{n}{r} \cdot \binom{r}{s}$$

with $n \geq r \geq s \geq 0$

Proof By the definition of the choice operation,

$$\begin{aligned}\binom{n-s}{r-s} \cdot \binom{n}{s} &= \frac{(n-s)!}{(r-s)!(n-r)!} \cdot \frac{n!}{s!(n-s)!} \\ &= \frac{n!}{(r-s)!(n-r)!s!} \\ &= \frac{n!r!}{r!(n-r)!s!(r-s)!} \\ &= \frac{n!}{r!(n-r)!} \cdot \frac{r!}{s!(r-s)!} \\ &= \binom{n}{r} \cdot \binom{r}{s}\end{aligned}$$

□

6.2. **Markov Matrices.** The following claim arose due to my studies of Voting Theory and the Schulze method.

Claim 6.2. ***(March 2, 2011)*** *A* $(n+p) \times (n+p)$ *Markov matrix* $M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$*, so that the upper left zero matrix is* $n \times n$*,* A *is* $n \times p$*,* B *is* $p \times n$ *and has the property that every entry is* $\frac{1}{n}$*, and the lower zero matrix is* $p \times p$*:*

1. *Has powers that are Markov matrices*
2. *Has positive even powers that are the same*
3. *Has positive odd powers that are the same, except possibly the first power*

Proof We have:

[5] Pascal's identity states:

$$\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$$

(1) Let M, N be $q \times q$ Markov matrices, so that

$$M = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q,1} & a_{q,2} & \cdots & a_{q,q} \end{bmatrix}$$

$$N = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,q} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q,1} & b_{q,2} & \cdots & b_{q,q} \end{bmatrix}$$

Then

$$M \cdot N = \begin{bmatrix} \sum_{j=1}^q a_{1,j}b_{j,1} & \sum_{j=1}^q a_{1,j}b_{j,2} & \cdots & \sum_{j=1}^q a_{1,j}b_{j,q} \\ \sum_{j=1}^q a_{2,j}b_{j,1} & \sum_{j=1}^q a_{2,j}b_{j,2} & \cdots & \sum_{j=1}^q a_{2,j}b_{j,q} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^q a_{q,j}b_{j,1} & \sum_{j=1}^q a_{q,j}b_{j,2} & \cdots & \sum_{j=1}^q a_{q,j}b_{j,q} \end{bmatrix}$$

If we sum each row we have the vector

$$\begin{bmatrix} \sum_{k=1}^q \sum_{j=1}^q a_{1,j}b_{j,k} \\ \sum_{k=1}^q \sum_{j=1}^q a_{2,j}b_{j,k} \\ \vdots \\ \sum_{k=1}^q \sum_{j=1}^q a_{q,j}b_{j,k} \end{bmatrix}$$

Since finite sums always converge, there is no issue exchanging the order of the sums (alternatively set $j = 1$, show that $a_{1,1}$ can be factored and the remaining sum in b is one, and proceed by cycling through all j), so we have the vector:

$$\begin{bmatrix} \sum_{j=1}^q \sum_{k=1}^q a_{1,j}b_{j,k} \\ \sum_{j=1}^q \sum_{k=1}^q a_{2,j}b_{j,k} \\ \vdots \\ \sum_{j=1}^q \sum_{k=1}^q a_{q,j}b_{j,k} \end{bmatrix}$$

or

$$\begin{bmatrix} \sum_{j=1}^q a_{1,j} \sum_{k=1}^q b_{j,k} \\ \sum_{j=1}^q a_{2,j} \sum_{k=1}^q b_{j,k} \\ \vdots \\ \sum_{j=1}^q a_{q,j} \sum_{k=1}^q b_{j,k} \end{bmatrix}$$

For any value of j, $\sum_{k=1}^q b_{j,k} = 1$, so in effect we have

$$\begin{bmatrix} \sum_{j=1}^q a_{1,j} \\ \sum_{j=1}^q a_{2,j} \\ \vdots \\ \sum_{j=1}^q a_{q,j} \end{bmatrix}$$

since we know the rows of M add up to one by hypothesis, we are left with a vector of size $q \times 1$ with entries all ones, as we wanted to show. Markov matrices are closed under matrix multiplication.

It follows that powers of a Markov matrix are Markov too, since we are multiplying Markov matrices by (the same) Markov matrix.

(2) The rather particular Markov matrix

$$M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

has second power:

$$M^2 = \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}$$

A remark: since M^2 is Markov, it follows that the $n \times n$ submatrix AB and the $p \times p$ submatrix BA are Markov too.

Now rewrite $B = \frac{1}{n}B^*$ so that $AB = A\frac{1}{n}B^*$ with B^* is a $p \times n$ matrix with entries all ones. Then AB^* adds the rows of A, which we know are equal to one, and $AB^* = C$ is a $n \times n$ matrix with entries all ones. Thus $AB = \frac{1}{n}C$.

Rewrite $BA = \frac{1}{n}B^*A$. Now let $B^*A = D$ is a $p \times p$ matrix with rows that are identical and sum the columns of A. As before we have that

$$M^2 = \frac{1}{n}\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$$

and

$$M^4 = \frac{1}{n^2}\begin{bmatrix} C^2 & 0 \\ 0 & D^2 \end{bmatrix}$$

Now, $C^2 = nC$ because of the property of C being all ones. We resort to a trick to show that $D^2 = nD$ as well, by writing out the explicit definition of D:

$$D = \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ a_1 & a_2 & \cdots & a_p \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_p \end{bmatrix}$$

and

$$D^2 = \begin{bmatrix} a_1 \sum_{i=1}^p a_i & a_2 \sum_{i=1}^p a_i & \cdots & a_p \sum_{i=1}^p a_i \\ \vdots & \vdots & \ddots & \vdots \\ a_1 \sum_{i=1}^p a_i & a_2 \sum_{i=1}^p a_i & \cdots & a_p \sum_{i=1}^p a_i \end{bmatrix}$$

or

$$D^2 = \left(\sum_{i=1}^p a_i\right) D$$

Careful, D is NOT Markov, so that the rows do not sum to 1, but D was generated summing the columns of A, a finite (non-Markov) matrix but with rows summing to 1, so that the sum operation in front of the D is asking us to do the double sum on the entries of A (sum all the entries of A). A is finite, so let's sum all the rows first and then all the columns (the order of the summing can be exchanged). Since all rows sum to 1, and there are n rows, it follows we are left with n as a result, and $D^2 = nD$.

[** From a patch-point-of-view, the statement that $D^2 = nD$, or $\left(\frac{1}{n}D\right)^2 = \frac{1}{n}D \Leftarrow N^2 = N$ with N is a Markov matrix with identical rows is analogous to the statement that $a(x) \star a(x) = a(x)$ with $a(x)$ is a *Pasquali patch*: at the "powering" level, the $\star$ operator as I defined it exchanges the integral summation and then adds continuously: the result of this is a uniform distribution (the analogue of the n in front of the D).**]

It is clear now that $M^4 = M^2$, and, as before, we use induction to show this is the case for all even powers of this particular matrix. So assume $M^{2k} = M^2$, and then $M^{2(k+1)} = M^{2k}M^2 = M^2M^2 = M^2$ and we are done.

(3) Since now we know that all even powers are the same, it follows that odd powers are the same too (except possibly the first power):

$$M^{2m+1} = M^{2m}M = M^2M = M^3$$

So all odd powers, except possibly M, look like M^3:

$$M^2M = \begin{bmatrix} \frac{1}{n}C & 0 \\ 0 & \frac{1}{n}D \end{bmatrix} \cdot \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{n}CA \\ \frac{1}{n}DB & 0 \end{bmatrix}$$

The product CA is a $(n \times n) \times (n \times p) = (n \times p)$ matrix with identical rows that adds the columns of A, call it D^*. On the other hand, the product $DB = D\frac{1}{n}B^*$ is a $(p \times p) \times (p \times n) = (p \times n)$ matrix that adds the rows of D which we already calculated must sum to n, so all its entries are such. Factor the n, and we get $DB = \frac{1}{n}nC^* = C^*$, with C^* all entries are ones. Finally, notice $\frac{1}{n}C^* = B$. We have:

$$M^3 = \begin{bmatrix} 0 & \frac{1}{n}D^* \\ B & 0 \end{bmatrix}$$

Since D^* was generated adding the columns of A (and A has entries that are nonzero at every row), one can think of several exceptions so that $\frac{1}{n}D^* \neq A$, and $M \neq M^3$. For example, in the special case where $n = p$, let A be the identity matrix; then $\frac{1}{n}D^*$ contains entries that are all $\frac{1}{n}$ and $\frac{1}{n}D^* \neq A \Rightarrow M \neq M^3$.

□

7. Proofs in Progress

Are convergent infinite sums of eigenvalues... descriptive of a finite function? Can we invent new, finite-dot-product functions by looking at convergent infinite sums of eigenvalues? Can we check equivalence of formulas by this method? For the claim where even and odd derivatives are constrained, what happens to periodic functions? Do they have the same invariant, sine and cosine? Sine and cosine plus a phase shift?

What is the relationship to the pythagorean theorem (can we prove it using this method)? Diophantine analysis with this method?

No reason why we should let surfaces be finite sums of x and y products.

Generalizing the construction to $\sum f_i(x)g_i(y)$

Eigenvalues for surfaces

Groups defined on patchixes or patches

All states are achievable.

Alternative ways of writing the star operator (similarity to convolution).

The derivative as progressive shape pathix changes using converging $\mathbb{P}$

Continuous-time patchix changes

How are zeroes transferred

Claim 1.X. If a collection $\mathbb{P}$ of *Pasquali patch* self-powers coincides with another $\mathbb{Q}$ in at least one element, then either $\mathbb{P} \subset \mathbb{Q}$ or $\mathbb{Q} \subset \mathbb{P}$ or both.

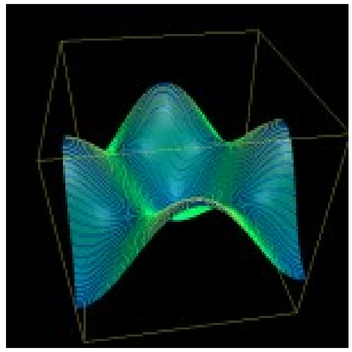

FIGURE 1. $p(x, y) = 1 - \cos(2\pi x)\cos(2\pi y)$

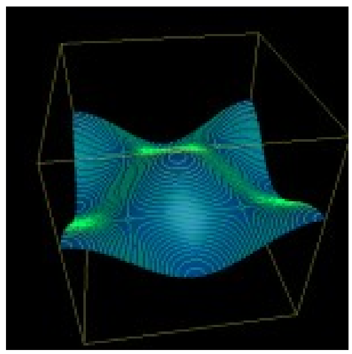

FIGURE 2. $p_2(x, y) = 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2}$

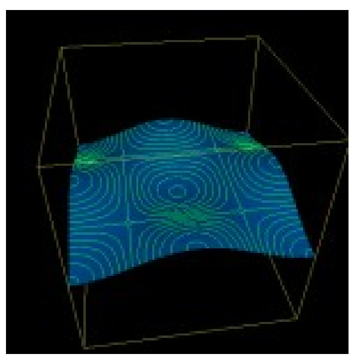

FIGURE 3. $p_3(x, y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{4}$

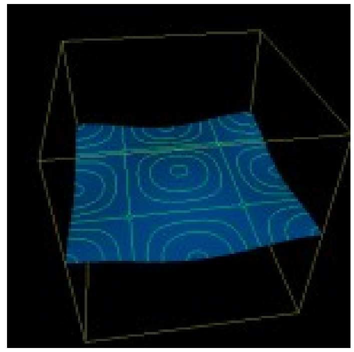

FIGURE 4. $p_4(x, y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{8}$

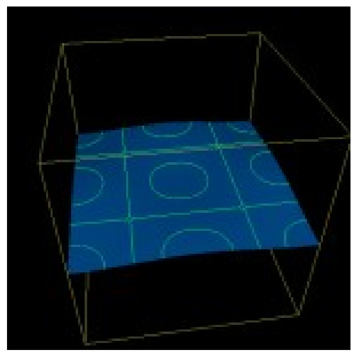

FIGURE 5. $p_5(x, y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{16}$

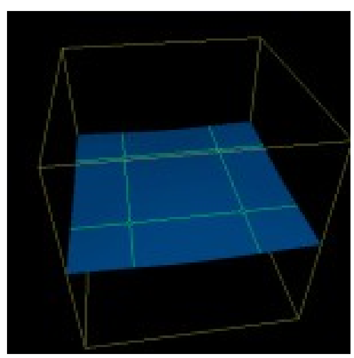

FIGURE 6. $p_6(x, y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{32}$