

COMPENDIUM OF CLAIMS AND PROOFS, INCLUDING NEW ONES, PART I

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1. OPERATIVE FRAMEWORK

Definition 1.1 (The Star and Circle Operators). (*October 17, 2010, January 17, 2013*)

On Two Surfaces

Let $f(x, y)$ and $g(x, y)$ be surfaces so that $f, g : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$. The star operator $\star : [0, 1]^2 \times [0, 1]^2 \rightarrow [0, 1]^2$ takes two surfaces and creates another in the following way:

$$(f(x, y), g(x, y)) \rightsquigarrow (f(1 - y, z), g(x, y)) \rightsquigarrow h(x, z) \rightsquigarrow h(x, y)$$

with the central transformation being defined by the integral

$$f(1 - y, z) \circ g(x, y) = \int_0^1 f(1 - y, z)g(x, y) dy = h(x, z)$$

and the last transformation that takes $h(x, z) \rightsquigarrow h(x, y)$ we will call $j : [0, 1]^2 \rightarrow [0, 1]^2$. Thus

$$f(x, y) \star g(x, y) = j(f(1 - y, z) \circ g(x, y)) = j\left(\int_0^1 f(1 - y, z)g(x, y) dy\right)$$

In the special case where $f(x, 0)$, we have

$$f(1 - y, 0) \circ g(x, y) = \int_0^1 f(1 - y, 0)g(x, y) dy = h(x, 0)$$

motivating the following:

On a Function and a Surface

Let $f(x)$ be a function $f : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$, and $g(x, y) : [0, 1]^2 \rightarrow \mathbb{R}^+ \cup \{0\}$. The star operator $\star : [0, 1] \times [0, 1]^2 \rightarrow [0, 1]$ takes the function and the surface and creates another function in the following way:

$$(f(x), g(x, y)) \rightsquigarrow (f(1 - y), g(x, y)) \rightsquigarrow h(x)$$

with the last transformation being defined by the integral

$$f(1 - y) \circ g(x, y) = \int_0^1 f(1 - y)g(x, y) dy = h(x)$$

Thus we have

$$f(x) \star g(x, y) = f(1 - y) \circ g(x, y) = \int_0^1 f(1 - y)g(x, y) dy$$

Claim 1.1. (*January 17, 2013*) Let $j : [0, 1]^2 \rightarrow [0, 1]^2$ carry $h(x, z) \rightsquigarrow h(x, y)$. Then:

$$\int_a^b j(h(x, z)) dx = j\left(\int_a^b h(x, z) dx\right)$$

Proof

$$\int_a^b j(h(x, z)) dx = \int_a^b h(x, y) dx = m(0, y)$$

On the other hand

$$j\left(\int_a^b h(x, z) dx\right) = j(m(0, z)) = m(0, y)$$

□

Definition 1.2 (*Pasquali patch*). (*April 22, 2010*) Define a continuous surface $p(x, y)$, with $p : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$, and let $\int_0^1 p(x, y) dx = 1$ be true regardless of the value of y . In other words, integrating such surface with respect to x yields the uniform probability distribution $u(y)$, $u : [0, 1] \rightarrow \{1\}$. We will call this a Pasquali patch.

Claim 1.2. (*October 10, 2010*) A well-behaved continuous probability distribution $b : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 b(x) dx = 1$, **star** a Pasquali patch, yields a continuous probability distribution $c : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 c(x) dx = 1$. In other words, a probability distribution $b(x)$ is taken to a probability distribution $c(x)$ via the Pasquali patch: $b(x) \star c(x)$.

Proof 1. Let $c(x) = b(x) \star p(x, y)$. We seek to show that

$$\int_0^1 c(x) dx = \int_0^1 b(x) \star p(x, y) dx = 1$$

Using **Definition 1.1**,

$$\int_0^1 b(x) \star p(x, y) dx = \int_0^1 \int_0^1 b(1-y)p(x, y) dy dx$$

Absolute convergence of the integrals allows us to exchange the order of integration (Fubini Theorem). Thus:

$$\int_0^1 \int_0^1 b(1-y)p(x, y) dx dy = \int_0^1 b(1-y) \int_0^1 p(x, y) dx dy$$

The innermost integral adds up to $u(y) = 1$ by **Definition 1.2**. Next

$$\int_0^1 b(1-y)u(y) dy = \int_0^1 b(1-y) \cdot 1 dy = 1$$

by virtue of $b(x)$ being a probability distribution. □

Proof 2. Via **Closure of Pasquali patches (Claim 1.4)**, a continuous probability distribution $b(x)$ can be thought of as $b(x, y)$ and a *Pasquali patch*, thus it **star** another *Pasquali patch* will yield a new *Pasquali patch* by closure of *Pasquali patches*. □

Example 1.1. (*October 10, 2010*) Let $b(x) = 6x(1-x)$. This is a beta(2,2) probability distribution and

$$\int_0^1 (6x(1-x)) dx = \int_0^1 (6x - 6x^2) dx = (3x^2 - 2x^3)|_0^1 = 1$$

is easily checked. Let $p(x, y) = x + \frac{1}{2}$ be a Pasquali patch, with

$$\int_0^1 \left(x + \frac{1}{2}\right) dx = \left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^1 = 1$$

for any choice of y . Then

$$b(x) \star p(x, y) = \int_0^1 b(1-y)p(x, y) dy = \int_0^1 6(1-y)(y) \left(x + \frac{1}{2}\right) dy = x + \frac{1}{2}$$

This probability distribution has already been shown to integrate to 1 with respect to x .

Corollary 1.3. (*October 10, 2010*) A continuous probability distribution $b : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$ *star* a Pasquali patch that solely a function of x yields a continuous probability distribution that is of the same form as the Pasquali patch. In other words, a probability distribution $b(x)$ is carried via the Pasquali patch $p(x, y) = p(x)$ to $p(x)$: $b(x) \rightsquigarrow p(x)$.

Proof 1. Using the definition of the star operator, $\int_0^1 b(1-y)p(x, y) dy = \int_0^1 b(1-y)p(x) dy = p(x)$. \square

Proof 2. Via **Claim 1.9**. A continuous probability distribution can be thought of as a *Pasquali patch*, and thus **Claim 1.9** applies. \square

Construction 1.1. (*October 10, 2010*) A way to construct a Pasquali patch is by positing

$$p(x, y) = f_1(x)g_1(y) + f_2(x) \frac{1 - g_1(y) \int_0^1 f_1(x) dx}{\int_0^1 f_2(x) dx} = f_1(x)g_1(y) + f_2(x) \frac{1 - g_1(y)F_1}{F_2}$$

for function choices $f_{1,2}(x)$, and $g_1(y)$.

Proof Since for *Pasquali patches* $\int_0^1 p(x, y) dx = 1$, it can be seen that such construction is a *Pasquali patch* because

$$\int_0^1 \left(f_1(x)g_1(y) + f_2(x) \frac{1 - g_1(y)F_1}{F_2} \right) dx = g_1(y)F_1 + \frac{F_2}{F_2} - \frac{g_1(y)F_1F_2}{F_2} = 1$$

regardless of function choices of y . \square

Claim 1.4 (Closure of *Pasquali patches*). (*October 12, 2010*) A Pasquali patch *star* a Pasquali patch yields a new Pasquali patch. In particular, a Pasquali patch *star itself* (*Pasquali patch powers*) will always yield another Pasquali patch.

Proof Take *Pasquali patches* $p(x, y)$ and $q(x, y)$. We want to show that $p(x, y) \star q(x, y) = r(x, y)$ is a *Pasquali patch*.

Thus, we want to show that $\int_0^1 r(x, y) dx = 1$. In other words,

$$\int_0^1 j \left(\int_0^1 p(1-y, z)q(x, y) dy \right) dx = 1$$

By **Claim 1.1**, we can write this as

$$j \left(\int_0^1 \int_0^1 p(1-y, z)q(x, y) dy dx \right)$$

We can exchange the order of integration because of absolute convergence of the integrals (Fubini Theorem). Thus

$$j \left(\int_0^1 p(1-y, z) \int_0^1 q(x, y) dx dy \right)$$

yields

$$j \left(\int_0^1 p(1-y, z)u(y) dy \right) = j \left(\int_0^1 p(1-y, z) dy \right)$$

This final integral evaluates to $j(u(z)) = u(y) = 1$ for any choice of y . Thus $\int_0^1 r(x, y) dx = 1$. \square

Example 1.2. (*October 12, 2010*) Take the Pasquali patch $p(x, y) = x^2y^3 + x \left(2 - \frac{2y^3}{3} \right)$. It is evident it is a Pasquali patch because

$$\int_0^1 \left[x^2y^3 + x \left(2 - \frac{2y^3}{3} \right) \right] dx = \left[\frac{x^3}{3}y^3 + \frac{x^2}{2} \left(2 - \frac{2y^3}{3} \right) \right] \Big|_0^1 = 1$$

We can calculate

$$p(1-y, z) = z^3y^2 - \frac{4z^3y}{3} + \frac{z^3}{3} - 2y + 2$$

so that the second “power” of the Pasquali patch is

$$p_2(x, y) = j \left(\int_0^1 p(1-y, z) p(x, y) dy \right) = j \left(\frac{29x}{15} + \frac{z^3 x}{90} + \frac{x^2}{10} - \frac{z^3 x^2}{60} \right)$$

and the final transformation j of the star operator gives

$$p_2(x, y) = \frac{29x}{15} + \frac{y^3 x}{90} + \frac{x^2}{10} - \frac{y^3 x^2}{60}$$

One easily checks $p_2(x, y)$ is indeed a Pasquali patch by performing the integral $\int_0^1 p_2(x, y) dx$ and ascertaining its equality to 1. The third Pasquali patch power is

$$p_3(x, y) = \frac{1741x}{900} - \frac{y^3 x}{5400} + \frac{59x^2}{600} + \frac{y^3 x^2}{3600}$$

Example 1.3. (October 15, 2010, October 17, 2010) The Pasquali patch

$$p(x, y) = 1 - \cos(2\pi x) \cos(2\pi y)$$

has powers:

$$p(x, y) = 1 - \cos(2\pi x) \cos(2\pi y)$$

$$p_2(x, y) = 1 + \frac{\cos(2\pi x) \cos(2\pi y)}{2}$$

$$p_3(x, y) = 1 - \frac{\cos(2\pi x) \cos(2\pi y)}{4}$$

$$p_4(x, y) = 1 + \frac{\cos(2\pi x) \cos(2\pi y)}{8}$$

$$p_5(x, y) = 1 - \frac{\cos(2\pi x) \cos(2\pi y)}{16}$$

$$p_6(x, y) = 1 + \frac{\cos(2\pi x) \cos(2\pi y)}{32}$$

⋮

$$p_n(x, y) = 1 - \frac{\cos(2\pi x) \cos(2\pi y)}{(-2)^{(n-1)}}$$

Proof by Induction We show that, by the inductive basis, $p(x, y) = 1 - \cos(2\pi x) \cos(2\pi y)$ must equal

$$p_1(x, y) = 1 - \frac{\cos(2\pi x) \cos(2\pi y)}{(-2)^{(0)}}$$

which a quick check shows is indeed the case.

Next, by the inductive step, we take as true that:

$$p_k(x, y) = 1 - \frac{\cos(2\pi x) \cos(2\pi y)}{(-2)^{(k-1)}}$$

Then,

$$\begin{aligned} p_{k+1}(x, y) &= j \left(\int_0^1 p_1(1-y, z) \cdot p_k(x, y) dy \right) \\ &= j \left(\int_0^1 (1 - \cos(2\pi(1-y)) \cos(2\pi z)) \cdot \left(1 - \frac{\cos(2\pi x) \cos(2\pi y)}{(-2)^{k-1}} \right) dy \right) \end{aligned}$$

The product of 1 with itself is 1, and such will integrate to 1 in the unit interval. So we save it. The integrals $\int_0^1 \cos(2\pi y) dy$ and $\int_0^1 \cos(2\pi - 2\pi y) dy$ both evaluate to zero, so we are left only with the task of evaluating the cross-term:

$$\begin{aligned} & \int_0^1 \cos(2\pi(1-y))\cos(2\pi z) \cdot \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{k-1}} dy \\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}} \int_0^1 \cos(2\pi - 2\pi y)\cos(2\pi y) dy \\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}} \int_0^1 \cos^2(2\pi y) dy \\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}} \cdot \frac{1}{2} \\ &= -\frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^k} \end{aligned}$$

Let's not forget the 1 we had saved, so:

$$p_{k+1}(x, y) = j \left(1 - \frac{\cos(2\pi x)\cos(2\pi z)}{(-2)^k} \right) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^k}$$

as we wanted to show. \square

Definition 1.3. *The limiting, steady-state, or stationary surface is a surface obtained by taking the limit as n approaches infinity of Pasquali patch powers $\lim_{n \rightarrow \infty} p_n(x, y)$, provided it exists, and we call it $p_\infty(x, y)$.*

Lemma 1.5. (October 17, 2010) *The stationary surface of*

$$p(x, y) = 1 - \cos(2\pi x) \cos(2\pi y)$$

is $p_\infty(x, y) = 1$.

Proof Since the *Pasquali patch* power collection of $p(x, y) = 1 - \cos(2\pi x) \cos(2\pi y)$ can be described by

$$p_n(x, y) = 1 - \frac{\cos(2\pi x) \cos(2\pi y)}{(-2)^{(n-1)}}$$

we need only take the limit as n approaches infinity of this formula:

$$\lim_{n \rightarrow \infty} p_n(x, y) = \lim_{n \rightarrow \infty} \left(1 - \frac{\cos(2\pi x) \cos(2\pi y)}{(-2)^{(n-1)}} \right) = 1$$

In this particular case, notice that $p_\infty(x, y) = 1$ is also a *Pasquali patch* because

$$\int_0^1 p_\infty(x, y) dx = 1$$

\square

Lemma 1.6. (December 3, 2012) *If the sequence $p_n(x, y)$ of Pasquali patch powers converge uniformly to $p_\infty(x, y)$, then $p_\infty(x, y)$ is a Pasquali patch.*

Proof We want to show that $\int_0^1 p_\infty(x, y) dx = 1$. First,

$$\int_0^1 p_\infty(x, y) dx = \int_0^1 \lim_{n \rightarrow \infty} p_n(x, y) dx$$

By uniform convergence of the sequence of *Pasquali patches*, we can exchange the order of the limit to obtain

$$\lim_{n \rightarrow \infty} \int_0^1 p_n(x, y) dx$$

Now for any n , $p_n(x, y)$ is a Pasquali patch, and therefore the integral is equal to 1 in every case (for any choice of n). Lastly,

$$\lim_{n \rightarrow \infty} \{1\}_n = 1$$

and we have arrived at what we wanted to show. \square

Claim 1.7 (Noncommutativity of the Star Operator). (*October 17, 2010*) *The star operator is noncommutative. Notice this is not restricted to Pasquali patches.*

Proof by Counterexample We want to show that $p(x, y) \star q(x, y) \neq q(x, y) \star p(x, y)$. Choose $p(x, y) = x$ and $q(x, y) = y$ without loss of generality. Then:

$$p(x, y) \star q(x, y) = p(x) \star q(x, y) = \int_0^1 p(1-y) \cdot q(x, y) dy = \int_0^1 (1-y)y dy = \int_0^1 (y-y^2) dy = \frac{1}{6}$$

and

$$q(x, y) \star p(x, y) = j \left(\int_0^1 q(1-y, z) \cdot p(x, y) dy \right) = j \left(\int_0^1 (z \cdot x) dy \right) = j(z \cdot x) = y \cdot x$$

are non-equal. \square

Claim 1.8 (Noncommutativity of *Pasquali patches* Under the Star Operator). (*October 17, 2010*) *We know in general the star operator is non-commutative, but we don't know that Pasquali patches as a subset of $[0, 1] \times [0, 1]$ surfaces ("Pasquali patchizes", due to their resembling matrixes in that they transform functions to other functions, as matrixes transform vectors to other vectors) don't commute.*

Proof by Counterexample Suppose the *Pasquali patches* $p(x, y) = x + \frac{1}{2}$ and $q(x, y) = 1 + xy - \frac{y}{2}$. Then

$$\begin{aligned} p(x, y) \star q(x, y) = p(x) \star q(x, y) &= \int_0^1 p(1-y) \cdot q(x, y) dy \\ &= \int_0^1 \left(\frac{3}{2} - y \right) \cdot \left(1 + xy - \frac{y}{2} \right) dy \\ &= \frac{5x}{12} + \frac{19}{24} \end{aligned}$$

where

$$\begin{aligned} q(x, y) \star p(x, y) &= j \left(\int_0^1 q(1-y, z) \cdot p(x, y) dy \right) \\ &= j \left(\int_0^1 q(1-y, z) \cdot p(x) dy \right) \\ &= j \left(p(x) \int_0^1 q(1-y, z) dy \right) \\ &= j(p(x) \cdot u(z)) = p(x) \\ &= x + \frac{1}{2} \end{aligned}$$

\square

Claim 1.9. (*October 17, 2010*) *A Pasquali patch star a Pasquali patch that is solely a function of x returns the second Pasquali patch.*

Proof Suppose the *Pasquali patches* $q(x, y)$ and $p(x, y) = p(x)$. Then:

$$\begin{aligned} q(x, y) \star p(x, y) &= j \left(\int_0^1 q(1-y, z) \cdot p(x) dy \right) \\ &= j \left(p(x) \int_0^1 q(1-y, z) dy \right) \\ &= j(p(x) \cdot u(z)) = p(x) \end{aligned}$$

\square

Claim 1.10 (Associativity of the Star Operator). (*October 17, 2010*) *The star operator is associative. Notice this is not restricted to Pasquali patches.*

Proof We want to show that $[p(x, y) \star q(x, y)] \star r(x, y) = p(x, y) \star [q(x, y) \star r(x, y)]$.

The LHS is:

$$\begin{aligned} [p(x, y) \star q(x, y)] \star r(x, y) &= j \left(\int_0^1 p(1-y, z) q(x, y) dy \right) \star r(x, y) \\ &= j \left(\int_0^1 \int_0^1 p(1-z, w) q(1-y, z) dz r(x, y) dy \right) \end{aligned}$$

$$= j \left(\int_0^1 \int_0^1 p(1-z, w) q(1-y, z) r(x, y) dz dy \right)$$

The RHS is:

$$\begin{aligned} p(x, y) \star [q(x, y) \star r(x, y)] &= p(x, y) \star j \left(\int_0^1 q(1-y, z) r(x, y) dy \right) \\ &= j \left(\int_0^1 p(1-z, w) \int_0^1 q(1-y, z) r(x, y) dy dz \right) \\ &= j \left(\int_0^1 \int_0^1 p(1-z, w) q(1-y, z) r(x, y) dy dz \right) \end{aligned}$$

We immediately see the equivalence using the Fubini Theorem to exchange the order of integration. \square

Corollary 1.11. (*October 17, 2010, December 3, 2012*) Pasquali patches inherit associativity.

Proof Pasquali patches are a subset of surfaces $[0, 1]^2 \rightarrow [0, 1]$ (Pasquali patchives), with the additional restriction that $\int_0^1 p(x, y) dx = 1$, and as such, inherit associativity. \square

Claim 1.12. (*December 3, 2012*) Pasquali patches that are functions of y (possibly x) explicitly generate by self-powering a countably infinite collection of Pasquali patches that are functions of y (possibly x) explicitly.

Proof by Induction Let $p(x, y)$ be a Pasquali patch that is a function of y (possibly x) explicitly. Thus the basis is true by hypothesis. By the inductive step, let $p_k(x, y)$, $k \in \mathbb{Z}_+$ be an explicit function of y (possibly x). Then:

$$p_{k+1}(x, y) = p_k(x, y) \star p(x, y) = j \left(\int_0^1 p_k(1-y, z) p(x, y) dy \right)$$

is an explicit function of y (possibly x) since $p_k(x, y)$ was an explicit function of y (possibly x), and the y was saved by the transformation to z in $p_k(1-y, z)$, so that the integral did not aggregate it. Then, the transformation j takes $z \rightsquigarrow y$ makes $p_{k+1}(x, y)$ an explicit function of y (possibly x). The result is a countable collection of explicit functions of y (possibly x) via self-powering, because $k \in \mathbb{Z}_+$. \square

Claim 1.13. (*December 3, 2012*) Pasquali patches that are functions of x alone (explicitly or otherwise) generate Pasquali patches that are the same as the original Pasquali patch via self-powering.

Proof by Induction $p(x) = p(x)$ by the identity of equality. $p_2(x) = p(x) \star p(x) = p(x)$ by **Claim 1.9**. This establishes the induction basis. Next, suppose $p_k(x) = p(x)$. Then $p_{k+1}(x) = p_k(x) \star p(x) = p(x) \star p(x) = p(x)$, and we are done. \square

Claim 1.14. (*December 3, 2012*) If the limiting surface $p_\infty(x, y)$ exists, it is NOT an explicit function of y . It is either an explicit function of x or constant for all x, y . We can describe it WLOG by $p_\infty(x)$.

Proof Case 1. Suppose we have a generator Pasquali patch $p(x)$ which is a function of x alone (explicitly or otherwise). Then, by **Claim 1.13**, $\forall n \in \mathbb{Z}_+$, $p_n(x) = p(x)$. Taking the limit as n approaches infinity, we obtain $\lim_{n \rightarrow \infty} p_n(x) = \lim_{n \rightarrow \infty} p(x)$, or $p_\infty(x) = p(x)$. This limiting surface is therefore also a Pasquali patch.

Case 2. In the space of uniformly convergent surfaces generated by Pasquali patch self-powers, the limiting surface will exist and will be a Pasquali patch itself by **Lemma 1.6**. Now suppose that $p_\infty(x, y)$ is an explicit function of y (possibly x). Then either it belongs to the collection of Pasquali patch powers generated by $p(x, y)$, or to a different collection altogether generated by Pasquali patch powers of, say, $q(x, y)$. If it belongs to the collection of Pasquali patch powers generated by $p(x, y)$, it must be equal to $p_n(x, y)$ for some n . But such is not an accumulation surface because we can generate $p_{n+1}(x, y)$ and onwards. It therefore must belong to a different collection, that generated by $q(x, y)$, and equals $q_m(x, y)$ for some m . The problem is that such isn't an accumulation surface either, being part of a (different) sequence

of *Pasquali patch* powers, and in a “parallel” collection. We are faced with a contradiction. $p_\infty(x, y)$ must therefore NOT be an explicit function of y : it is either explicitly or not a function of x , that is, $p_\infty(x)$. Recall it is also a *Pasquali patch* by **Lemma 1.6**. \square

1.1. Mechanics of Powering Proofs.

Claim 1.15 (Powering Symmetry). (*December 30, 2012*) Let the n th Pasquali patch power of $p(x, y)$, $p_n(x, y)$ be denoted by P^n . Then for $m, n \geq 1$, $P^n \star P^m = P^m \star P^n$.

Proof by Induction By definition of powering,

$$P^1 \star P^m = P^1 \star \underbrace{(P^1 \star \dots \star P^1)}_{m \text{ times}}$$

Then, by **Claim 1.10 Associativity of the Star Operator**, this is equivalent to

$$\underbrace{(P^1 \star \dots \star P^1)}_{m \text{ times}} \star P^1 = P^m \star P^1$$

Next assume that, for fixed $k, m \geq 1$, $P^k \star P^m = P^m \star P^k$.

Then $P^{k+1} \star P^m = (P^1 \star P^k) \star P^m$, where this bit we take as the definition of powering, and then, by **Claim 1.10** again, $P^1 \star (P^k \star P^m)$, which then by our inductive hypothesis equals $P^1 \star (P^m \star P^k)$ and this is $(P^m \star P^k) \star P^1$ by the inductive basis. Again, **Claim 1.10** gives $P^m \star (P^k \star P^1)$ and lastly $P^m \star P^{1+k}$ by definition of powering. Axiomatic commutativity of the positive integers gives $P^m \star P^{k+1}$ and we are done.

By symmetry of equality itself, we need only do a single proof of induction on one of the power parameters. \square

Claim 1.16. (*December 30, 2012*) $P^n \star P^m = P^{m+n}$.

Proof by Induction First, fix $m \geq 1$. By the definition of powering, $P^1 \star P^m = P^{m+1}$. Next, suppose it's true that $P^k \star P^m = P^{m+k}$. Then $P^{k+1} \star P^m$ equals $(P^1 \star P^k) \star P^m$ by definition of powering, and by **Claim 1.10 Associativity of the Star Operator** we get $P^1 \star (P^k \star P^m)$ or $P^1 \star P^{m+k}$ using the inductive hypothesis. By the inductive basis, this becomes $P^{(m+k)+1}$, that is $P^{m+(k+1)}$ using associativity of the positive integers, which we take as axiomatic.

Having done so, next fix $n \geq 1$ and let m vary. We have $P^n \star P^1 = P^1 \star P^n$ by **Claim 1.15 Powering Symmetry**. Since this part of the proof is identical to the one we just wrote, we are done. \square

Remark 1.1. (*April 22, 2010*) *Probabilistic Interpretation.* In direct analogy to the Chapman-Kolmogorov equation, transition from state $y \in [0, 1]$ to $x \in [0, 1]$ in $n + m$ steps can be achieved by first transitioning to an intermediate state x° in n steps and then jumping from there to x in m more steps. Particularly, all states are achievable from a starter state since for Pasquali patch $p(x, y)$ there are only a finite number of zeroes in the domain because the surface is well-behaved.

2. FIXED DISTRIBUTION CONJECTURE AND PROOFS

Conjecture 2.1. (*December 3, 2012*) It is only natural to ask if there exists a (bounded, well-behaved, probability) function $a(x)$, $a : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$ that remains fixed when starred by a Pasquali patch $p(x, y)$. In other words, is there an $a(x)$ so that $a(x) \star p(x, y) = a(x)$?

Claim 2.1. (*December 3, 2012*) Suppose **Conjecture 2.1** is true. Then $a(x)$ remains fixed for all Pasquali patch powers generated by $p(x, y)$.

Proof by Induction We take as true that $a(x) \star p(x, y) = a(x)$ by **Conjecture 2.1**. By the inductive step, we have that $a(x) \star p_k(x, y) = a(x)$. Then:

$$a(x) \star p_{k+1}(x, y) = a(x) \star (p_k(x, y) \star p(x, y))$$

By **Claim 1.10 Associativity of the Star Operator**, this last expression we can write as:

$$(a(x) \star p_k(x, y)) \star p(x, y) = a(x) \star p(x, y) = a(x)$$

and we are done. \square

Claim 2.2. (December 3, 2012) *The only function $b(x)$ that makes true the expression $a(x) \star b(x) = a(x)$, $a(x)$ is a probability distribution $a : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 a(x) = 1$ (and therefore a Pasquali patch from another viewpoint), is $a(x)$ itself.*

Proof 1 We resort to the definition of the star operator, and the LHS is

$$a(x) \star b(x) = \int_0^1 a(1-y)b(x) dy = b(x) \int_0^1 a(1-y) dy = b(x)$$

On the other hand, the RHS is $a(x)$, and $b(x) = a(x)$. \square

Proof 2 By **Claim 1.13**, the LHS is $b(x)$. The RHS is $a(x)$ and the equality is established. \square

Corollary 2.3. (December 3, 2012) $a(x) \star a(x) = a(x)$

Proof By **Claim 2.2**, since the only function of x that makes the expression true is $a(x)$, the result follows by direct plugging-in to the original expression. \square

Corollary 2.4. (December 3, 2012) $a(x)$ is fixed for the limiting Pasquali patch probability distribution of some collection of uniformly convergent Pasquali patch self-powers of $p(x, y)$ if and only if $p_\infty(x) = a(x)$.

Proof \Rightarrow By **Claim 1.14**, the Pasquali patch self-powers of $p(x, y)$ converge to $p_\infty(x)$, a Pasquali patch (by **Lemma 1.6**) which is NOT explicitly a function of y . By hypothesis, $a(x) \star p_\infty(x) = a(x)$. Then by **Claim 1.9**, $a(x) \star p_\infty(x) = p_\infty(x)$. It follows that $p_\infty(x) = a(x)$.

\Leftarrow By hypothesis, $p_\infty(x) = a(x)$, and by **Corollary 2.3**, $a(x) \star a(x) = a(x)$. Plugging in the first equation with the second at the appropriate location yields the desired result, and $a(x) \star p_\infty(x) = a(x)$. \square

Corollary 2.5. (December 3, 2012) $a(x)$ is fixed for a collection of (uniformly convergent) self-powers of the Pasquali patch $p(x, y)$ if and only if it is fixed for the limiting Pasquali patch (by **Lemma 1.6**) $p_\infty(x)$.

Proof \Rightarrow By hypothesis, $a(x)$ is fixed for all (uniformly convergent) $p_n(x, y)$. Since they are uniformly convergent, they converge to $p_\infty(x)$. Now, $a(x) \star a(x) = a(x)$ is true by **Corollary 2.3**, and $a(x)$ is the ONLY (probability) probability distribution or Pasquali patch that makes the statement true by **Claim 2.2**. Also, $p_\infty(x) \star p_\infty(x) = p_\infty(x)$ by **Lemma 1.6** ($p_\infty(x)$ is a Pasquali patch) and **Claim 1.13**. It follows that $p_\infty(x) = a(x)$ with these two pieces of information. Lastly, by **Corollary 2.4** (\Leftarrow), we have that $a(x) \star p_\infty(x) = a(x)$.

Indirect Proof. \Leftarrow Suppose otherwise, that $a(x) \star p_\infty(x) = a(x)$ and $p_\infty(x) = a(x)$ by **Corollary 2.4**, which in turn implies of **Corollary 2.3** that $a(x) \star a(x) = a(x) \Rightarrow p_\infty(x) \star p_\infty(x) = p_\infty(x)$, but $a(x) \star p(x, y) \neq a(x)$ (for some uniformly convergent $p(x, y)$). Then:

$$(a(x) \star p(x, y)) \star a(x) \neq a(x) \star a(x)$$

$$a(x) \star (p(x, y) \star a(x)) \neq a(x) \star a(x)$$

by **Claim 1.10 Associativity of the Star Operator**

$$a(x) \star (a(x)) \neq a(x)$$

by **Claim 1.13** and

$$p_\infty(x) \star p_\infty(x) \neq p_\infty(x)$$

contradicts the last implication of the hypothesis. \square

Remark 2.1 (Guiding Maplet). **(December 7, 2012)**

The following maplet shows at a high-level the most important claims that are relevant to practically calculating the limiting Pasquali patch.

Part I: (graph here)

Part II: (graph here)

Remark 2.2. (*December 8, 2012*) Suppose that, by some sorcery or heuristics or good guess, we have found a candidate probability distribution $a(x)$ so that it is fixed for a given Pasquali patch $p(x, y)$. We can hazard the very good guess that the stationary limiting surface is $p_\infty(x) = a(x)$, and this itself is a Pasquali patch. It is only a “very good guess” because we have shown this equation to be true for Pasquali patch power collections that converge uniformly. In order to be completely certain that $p_\infty(x) = a(x)$, we would have to show that the Pasquali patch power collection generated by $p(x, y)$ possesses this property, a task left to, for example, a Weierstrass M -test. The property of uniform convergence becomes unduly burdensome (imagine having to show for each $p(x, y)$ uniform convergence before we can conclude without a shadow of a doubt that the limiting Pasquali patch is $p_\infty(x) = a(x)$).

We are left with two choices: either (1) we prove for **Construction 1.1** or some-such family of Pasquali patches the property of uniform convergence, thus limiting ourselves to the study of these particular Pasquali patches (seems overly restrictive from my viewpoint), or (2) we weaken the uniform convergence criterion to just convergence (of any sort). In essence, this second argument implies showing that the limit of the integrals of converging Pasquali patch powers generated by $p(x, y)$ equals the integral of the limiting Pasquali patch: a revision of **Lemma 1.6** (and subsequent claims that use it). With this, we need only posit convergence and $p_\infty(x) = a(x)$ automatically. This second approach is the one I’m most inclined for and working toward.

Lemma 2.6. (*December 3, 2012*) If the sequence $p_n(x, y)$ of Pasquali patch powers converge to $p_\infty(x, y)$, then $p_\infty(x, y)$ is a Pasquali patch.

Proof Suppose we have a collection of converging Pasquali patch powers generated by the Pasquali patch $p(x, y)$. They must converge to something, so call this $p_\infty(x, y)$. Next look at the sequence

$$\left\{ \int_0^1 p_n(x, y) dx \right\}_{n \in \mathbb{Z}_+} = \{1\}_n$$

The sequence is obviously bounded above, but in particular, the least upper bound b is 1. Similarly, the infimum is itself 1. Next let’s look at $\int_0^1 p_\infty(x, y) dx = c$. Suppose that $c > b$. Then there is an integral of some Pasquali patch power that must be greater than 1. But this is a contradiction, since all Pasquali patch power integrals are 1. Next suppose $c < b$. The issue is that the sequence $\{1\}_n$ is never less than 1 either. Clearly $c = b$ and we are done. \square

Things that change with this amendment:

- Proof of **Claim 1.14**. Rather than having Case 2 be the space of convergent surfaces, it is for convergent surfaces. Also, $p_\infty(x, y)$ is a Pasquali patch by **Lemma 2.6**.
- **Corollary 2.4**. Rather than being for convergent Pasquali patches, the corollary holds for convergent Pasquali patches: The proof holds with **Lemma 2.6**. This we shall call:

Corollary 2.7.

- **Corollary 2.5** Again, the corollary holds for convergent Pasquali patches and not just convergent ones. The proof holds with **Lemma 2.6**. This we shall call:

Corollary 2.8.

- **Guiding Maplet 2.1**. Wherever there is a convergence, we can substitute just “convergence.” Wherever there is a **Lemma 1.6**, we can substitute it by **Lemma 2.6**. Wherever there is a **Corollary 2.4**, we can change it to **Corollary 2.7**, and wherever there is a **Corollary 2.5**, we can substitute it to **Corollary 2.8**.

Remark 2.3. (*December 16, 2012*) This is why **Example 1.3** actually has a limiting probability distribution. The Pasquali patch power collection does not converge uniformly (it converges at the nodes first, e.g.), and yet it stabilizes to the uniform Pasquali patch eventually.

A corollary to all this is:

Corollary 2.9. (*December 16, 2012*) $\exists a(x)$ so that $a(x) \star p(x, y) = a(x)$ if and only if the powers of $p(x, y)$ converge.

Proof \Rightarrow Suppose not, that the powers $p(x, y)$ do not converge. One way for this to happen and have $\int_0^1 p(x, y) dx = 1$ is if the powers get stuck, for example, in a loop, so that, say,

$$p_{2n+1}(x, y) = p(x, y)$$

and

$$p_{2n}(x, y) = q(x, y)$$

and

$$p(x, y) \neq q(x, y)$$

Then by hypothesis $a(x) \star p_{2n+1}(x, y) = a(x)$ and it must also be true that $a(x) \star p_{2n}(x, y) = a(x)$ for this same $a(x)$ (**Claim 2.1**). This implies that $a(x) \star p(x, y) = a(x)$ and $a(x) \star q(x, y) = a(x)$. Now each even powers and odd powers are independently convergent, $p_{2n+1}(x, y)$ to $p(x, y)$ and $p_{2n}(x, y)$ to $q(x, y)$, which implies that $a(x) = p(x, y)$ in the first case and $a(x) = q(x, y)$ in the second case (**Corollary 2.7**). Other than the fact that both $p(x, y)$ and $q(x, y)$ could be functions of x, y , they are defined to be unequal to each other. We have a contradiction.

Next, let's make the loop larger. Suppose that the powers oscillate in k different surface forms, so that we have $p^1(x, y) \dots p^k(x, y)$ surfaces after which we return to the original. The above would mean that $p_n(x, y) = p^{n \bmod k}(x, y)$. Each "in-between" power is independently convergent to itself after k more powers, and $a(x) = p^k(x, y)$ due to **Corollary 2.7**. But these surfaces were defined to be different from each other, and so, as before, we face a contradiction.

We show of course the $k + 1$ case. Suppose the powers oscillate in $k + 1$ different surface forms, with $p^1(x, y) \dots p^{k+1}(x, y)$ surfaces after which we return to the original. Then

$$p_n(x, y) = p^{n \bmod (k+1)}(x, y)$$

with each in-between power independently convergent to itself after $k+1$ more powers, and $a(x) = p^{k+1}(x, y)$ due to **Corollary 2.7** each. Each of these surfaces was different. We have a contradiction.

Now we can make the periodicity as large as we like.

This same argument applies for the case in which the oscillatory pattern is established (independently) to m different surfaces eventually. Suppose that we have $p^1(x, y) \dots p^m(x, y)$ surfaces toward which each periodic sequence $p_{n \bmod m}(x, y)$ converges eventually, with

$$\lim_{n \rightarrow \infty} p_{n \bmod m}(x, y) = p^{n \bmod m}(x, y)$$

An argument with respect to $a(x)$ and the surfaces of convergence being unequal leads us to contradict, even inductively, that such a scenario is a possibility. An eventual convergent loop cannot form, no matter how large the loop or how slow or fast each convergence.

We have shown essentially that we cannot have an $a(x)$ and have the *Pasquali patch* powers of $p(x, y)$ oscillate in any shape or form in the long run.

We have not shown the "divergent" situation, in which the value of at least one point $p_n(\bar{x}, \bar{y}) \rightarrow \infty$. In particular, the product $a(1 - \bar{y}) \cdot p_n(\bar{x}, \bar{y}) \rightarrow \infty$, and the value of the integral $\int_0^1 a(1 - y) \cdot p_n(\bar{x}, y) dy \rightarrow \infty$ as well. Thus $a(\bar{x})$ diverges and $a(x)$ was never bounded or well-behaved (Contradicting **Conjecture 2.1** regarding $a(x)$). NB: This argument can probably be made more rigorous.

Lastly, since the powers neither stabilize oscillatorily nor can they diverge, they must converge.

(**December 24, 2012**) \Leftarrow Since the powers of $p(x, y)$ converge, they converge to $p_\infty(x)$ by **Claim 1.14**, a function of x alone or constant for all x, y . Then it is certainly true that $p(x, y) \star p_\infty(x) = p_\infty(x)$ by **Claim 1.9**. It must also be true by the same claim that $p_\infty(x) \star p_\infty(x) = p_\infty(x)$. If we substitute one equation into the other, we get $p_\infty(x) \star (p(x, y) \star p_\infty(x)) = p_\infty(x)$. By **Claim 1.10, Associativity of the Star Operator**, we can rewrite that as $(p_\infty(x) \star p(x, y)) \star p_\infty(x) = p_\infty(x)$. But now the arguments in parenthesis must be equal to $p_\infty(x)$, in other words, $p_\infty(x) \star p(x, y) = p_\infty(x)$. Thus $p_\infty(x)$ is fixed for $p(x, y)$. Let $a(x)$ be exactly this $p_\infty(x)$ and we are done. \square

Remark 2.4. (*December 24, 2012*) It is now evident that, so long as we can find a fixed $a(x)$, we know that the powers of $p(x, y)$ (1) converge, (2) the stationary surface is $p_\infty(x)$, that (3) $a(x) = p_\infty(x)$, and (4) $a(x)$ is fixed for all powers, including the stationary surface itself. Conversely, if we know that $p_\infty(x)$ for some $p(x, y)$, then such is in fact $a(x)$. Our efforts from now on should focus on establishing a mechanism to find such $a(x)$.

3. FIXED DISTRIBUTION EXISTENCE AND PROOFS

Claim 3.1. (*December 2, 2012*) For **Construction 1.1**, $a(x)$ exists provided $B = a(x) \star g_1(y)$ converges and can be solved. Its explicit form is

$$a(x) = p_\infty(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x) \right) B$$

Proof We are looking for $a(x)$ so that $a(x) \star p(x, y) = a(x)$, with

$$p(x, y) = f_1(x)g_1(y) + f_2(x) \left(\frac{1 - g_1(y)F_1}{F_2} \right)$$

Using the definition of the star operator (**Definition 1.1**), this is

$$\int_0^1 a(1-y) \left(f_1(x)g_1(y) + f_2(x) \left(\frac{1 - g_1(y)F_1}{F_2} \right) \right) dy = a(x)$$

Expansion results in:

$$a(x) = f_1(x) \int_0^1 a(1-y)g_1(y) dy + \frac{f_2(x)}{F_2} \cdot 1 - \frac{F_1}{F_2} f_2(x) \int_0^1 a(1-y)g_1(y) dy$$

where we have simplified $\int_0^1 a(1-y) dy$ to 1 because the transformation to the y-axis does not change the integral result (it remains a probability distribution).

Rearranging, we have

$$\frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x) \right) \int_0^1 a(1-y)g_1(y) dy$$

or

$$a(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x) \right) B = p_\infty(x)$$

(**Remark 2.4**) and derivatives

$$a^i(x) = \frac{f_2^i(x)}{F_2} - \left(\frac{f_2^i(x)F_1}{F_2} - f_1^i(x) \right) B$$

We want to obtain B , but the expression $\int_0^1 a(1-y)g_1(y) dy$ has to be clearly defined. We use the tabular method to simplify the integration by parts.

Derivatives	Integrals
$a(1-y)$	$g_1(y)$
$-a'(1-y)$	$G_1^1(y)$
$a''(1-y)$	$G_1^2(y)$
\vdots	\vdots

Viewed from a different vantage-point, we could have

Derivatives	Integrals
$g_1(y)$	$a(1-y)$
$g_1'(y)$	$-A^1(1-y)$
$g_1''(y)$	$A^2(1-y)$
\vdots	\vdots

Lastly, we have:

$$B = a(1-y)G_1^1(y) + a'(1-y)G_1^2(y) + \dots \Big|_0^1 = \sum_{i=0}^{\infty} a^i(1-y)G_1^{i+1}(y) \Big|_0^1$$

or

$$B = -g_1(y)A^1(1-y) - g_1'(y)A^2(1-y) - \dots \Big|_0^1 = -\sum_{i=0}^{\infty} g_1^i(y)A^{i+1}(1-y) \Big|_0^1$$

If the sum diverges we are stuck, but if the sum converges we are good. \square

Corollary 3.2. (*December 25, 2012*) Pasquali patches constructed as by **Construction 1.1** with (finite) polynomial function choices for $f_1(x)$ and $f_2(x)$ are guaranteed to have a fixed $a(x) = p_{\infty}(x)$ regardless of (integrable) function choice $g_1(y)$. Similarly, **Construction 1.1** Pasquali patches with a (finite) polynomial choice for $g_1(y)$ are guaranteed to have such fixed $a(x)$ as well, regardless of (integrable) function choices for $f_1(x)$ and $f_2(x)$.

Proof Since

$$B = a(1-y)G_1^1(y) + a'(1-y)G_1^2(y) + \dots \Big|_0^1 = \sum_{i=0}^{\infty} a^i(1-y)G_1^{i+1}(y) \Big|_0^1$$

and the derivatives of $a(x)$ are eventually zero (for all subsequent derivatives), the sum itself is finite. Thus, B converges, which implies that **Claim 3.1** applies.

Next, since

$$B = -g_1(y)A^1(1-y) - g_1'(y)A^2(1-y) - \dots \Big|_0^1 = -\sum_{i=0}^{\infty} g_1^i(y)A^{i+1}(1-y) \Big|_0^1$$

and the derivatives of $g_1(y)$ are eventually zero (including all subsequent derivatives), such sum is also finite and B converges. Again, **Claim 3.1** applies. \square

Example 3.1. (*January 16, 2011*) In **Example 1.2**, we had the Pasquali patch

$$p(x, y) = x^2 y^3 + x \left(2 - \frac{2y^3}{3} \right)$$

with $f_1(x) = x^2$, $f_2(x) = 2x$, $g_1(y) = y^3$. This implies by **Claim 3.1** that

$$a(x) = p_{\infty}(x) = 2x - \left(\frac{2x}{3} - x^2 \right) B$$

with derivatives

$$a'(x) = 2 - \left(\frac{2}{3} - 2x \right) B$$

$$a''(x) = -2B$$

Specifically,

$$\begin{aligned} a(1) &= 2 + \frac{B}{3} & \text{and} & & a(0) &= 0 \\ a'(1) &= 2 + \frac{4B}{3} & \text{and} & & a'(0) &= 2 - \frac{2B}{3} \\ a''(1) &= -2B & \text{and} & & a''(0) &= -2B \end{aligned}$$

Next, we want to calculate B :

$$B = \cancel{a(0)G_1^1(1)} + a'(0)G_1^2(1) + a''(0)G_1^3(1) - \left(\cancel{a(1)G_1^1(0)} + \cancel{a'(1)G_1^2(0)} + \cancel{a''(1)G_1^3(0)} \right)$$

The parenthetical part dies because integrals of y^3 evaluated at 0 vanish. So does the first term since $a(0) = 0$. So we are left with:

$$B = a'(0)G_1^2(1) + a''(0)G_1^3(1) = \frac{6-2B}{3} \cdot \frac{1}{20} + -2B \cdot \frac{1}{120}$$

which solves

$$B = \frac{6}{63} = \frac{2}{21}$$

Thus, we have that

$$a(x) = p_\infty(x) = 2x - \left(\frac{2x}{3} - x^2\right) \frac{2}{21} = \frac{2x^2}{21} + \frac{122x}{63}$$

For a consistency check, by **Lemma 2.6** this is a Pasquali patch and therefore

$$\int_0^1 \left(\frac{2x^2}{21} + \frac{122x}{63}\right) dx = 1$$

can be verified to indeed be the case.

Remark 3.1. Practically speaking, all functions $f(x), g(y)$ with Taylor polynomial representations in the domain $[0, 1]$ will converge.

4. PROBABILITY DISTRIBUTION TRANSFORMATIONS VIA PASQUALI PATCH POWER COLLECTIONS

Claim 4.1. (January 12, 2013) Take a Pasquali patch power collection union its limiting surface, $\mathbb{P} \cup p_\infty(x) = \mathbb{P}^\infty$, and a well-behaved, bounded probability distribution $c(x)$, with $c : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$, and $\int_0^1 c(x) dx = 1$. Then $c(x) \star \mathbb{P}^\infty = \mathbb{C}$ is a collection of transformed probability distributions of x (explicit or not).

Proof by Induction Case 1: the generator of \mathbb{P}^∞ is a *Pasquali patch* function of x alone (explicitly or implicitly). Then, by **Claim 1.13** all *Pasquali patches* in the collection are functions of x (explicitly or implicitly), and in fact all powers in \mathbb{P} equal $p(x)$, including $p_\infty(x)$ (**Case 1 of Claim 1.14**). Begin with

$$c(x) \star P^1 = c(x) \star p(x) = p(x)$$

by **Claim 1.9**. Assume that $c(x) \star P^k = p(x)$. Then

$$c(x) \star P^{k+1} = c(x) \star (P^1 \star P^k) = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star p(x) = p(x) \star p(x) = p(x)$$

We used **Claim 1.15 Powering Symmetry** to exchange the order of the powers, **Claim 1.10 Associativity of the Star Operator**, and again by **Claim 1.9** the last equality holds. Furthermore, since $p_\infty(x) = p(x)$, $c(x) \star p_\infty(x) = c(x) \star p(x) = p(x)$. This last part is justified by **Claim 1.9** once more. Thus, $c(x) \star \mathbb{P}^\infty = \mathbb{C} = \{p(x)\}$ and all functions are explicit or implicit functions of x as we wanted to show. By **Claim 1.2**, the collection is made up of probability distributions on $[0, 1]$.

Case 2: the generator of \mathbb{P}^∞ is a *Pasquali patch* function of y explicitly. Begin by

$$c(x) \star P^1 = \int_0^1 c(1-y)p(x, y) dy = c_1(x)$$

Since we are integrating with respect to y we can see that this function $c_1(x)$ is either explicitly or implicitly a function of x . Next assume that $c(x) \star P^k = c_k(x)$ is an explicit or implicit function of x . Then

$$c(x) \star P^{k+1} = c(x) \star P^{1+k} = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star P^1 = c_k(x) \star P^1$$

This last part is $\int_0^1 c_k(1-y)p(x, y) dy$, which, by virtue of integrating in terms of y yields an explicit or implicit function of x . Now take $c(x) \star p_\infty(x) = p_\infty(x)$ by **Claims 1.14** and **1.9**. Thus, all functions in the collection $c(x) \star \mathbb{P}^\infty = \mathbb{C}$ are explicit or implicit functions of x . Finally, by **Claim 1.2**, the collection is made up of probability distributions on $[0, 1]$. \square

Claim 4.2. (January 12, 2013) Let $c(x) \star P^m = c_m(x)$, the m th transform of $c(x)$ via the m th power of $p(x, y)$. An equivalent way to obtain the m th transform is by $c_{m-1}(x) \star P^1 = c_m(x)$.

Proof by Induction Begin by $c(x) \star P^1 = c_1(x)$ using the first statement and $c_0(x) \star P^1 = c_k(x)$ using the second statement. Clearly the two statements are equivalent. Next, assume that the statement holds for the k th transform, so that $c(x) \star P^k = c_k(x)$ and $c_{k-1}(x) \star P^1 = c_k(x)$. Now $c(x) \star P^{k+1} = c_{k+1}(x)$ by definition. An equivalent way to write this is

$$c(x) \star P^{1+k} = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star P^1 = c_k(x) \star P^1$$

Here we have used **Claim 1.10 Associativity of the Star Operator**. Thus $c_k(x) \star P^1 = c_{k+1}(x)$, and we are done. \square

Claim 4.3. (January 13, 2013) *The collection \mathbb{C} converges to $p_\infty(x)$.*

Indirect Proof 1 Suppose $\lim_{m \rightarrow \infty} c_m(x) \neq p_\infty(x)$. Then

$$\left[\lim_{m \rightarrow \infty} c_m(x) \right] \star p_\infty(x) \neq p_\infty(x) \star p_\infty(x)$$

and

$$p_\infty(x) \neq p_\infty(x) \star p_\infty(x)$$

is a contradiction of **Claim 1.13** or **Claim 1.9**. Thus $\lim_{m \rightarrow \infty} c_m(x) = p_\infty(x)$. \square

Indirect Proof 2 Suppose that the sequence of probability distributions $c_m(x)$ diverge, that

$$\lim_{m \rightarrow \infty} c_m(\bar{x}) = \infty$$

for some \bar{x} . This would have to mean that $\lim_{m \rightarrow \infty} \int_0^1 c(1-y)p_m(\bar{x}, y) dy = \infty$. But then the sequence $p_m(\bar{x}, y)$ would have to be divergent, because $c(x)$ is chosen to be well-behaved and bounded. We have a contradiction, because the sequence $p_m(x, y)$ is well-behaved and bounded by virtue of being *Pasquali patch* powers with bounded first-power generator and fixed volume. Thus $\lim_{m \rightarrow \infty} c_m(x)$ converges.

Next suppose the sequence of probability distributions $c_m(x)$ does not converge to $p_\infty(x)$ but to some other probability distribution in the collection \mathbb{C} , say $c_k(x)$. The issue is this isn't an accumulation probability distribution, since we can generate $c_{k+1}(x)$ by starring by P^1 on the right (**Claim 4.2**). Lastly, say we choose a probability distribution outside of the collection \mathbb{C} , in some other collection \mathbb{D} . If we choose $d_k(x)$, this isn't an accumulation probability distribution for \mathbb{C} because it was generated by *Pasquali patch* $q(x, y)$, and also $d_{k+1}(x)$ can be generated by right-starring by Q^1 . Then pick $p_\infty^d(x)$, the limiting probability distribution in \mathbb{D} . There's just no way to tie such to the generator collection \mathbb{P}^∞ , since it is an accumulation probability distribution for \mathbb{D} , not for \mathbb{C} . The only choice left (that makes sense) is $p_\infty^c(x)$. \square

Corollary 4.4. (January 13, 2013)

$$\lim_{m \rightarrow \infty} c_m(x) = p_\infty(x)$$

if and only if

$$\lim_{m \rightarrow \infty} [c(x) \star P^m] = c(x) \star \left[\lim_{m \rightarrow \infty} P^m \right]$$

In other words, we can pull the limiting process under the integral.

Proof \Rightarrow By definition, $c(x) \star P^m = c_m(x)$. Then

$$\lim_{m \rightarrow \infty} c_m(x) = \lim_{m \rightarrow \infty} [c(x) \star P^m] = p_\infty(x)$$

by hypothesis.

Next,

$$\lim_{m \rightarrow \infty} [c(x) \star P^m] = \lim_{m \rightarrow \infty} \int_0^1 c(1-y)p_m(x, y) dy = p_\infty(x)$$

An equivalent expression can be found through **Claim 1.9**:

$$p_\infty(x) = \int_0^1 c(1-y)p_\infty(x) dy$$

This is,

$$p_\infty(x) = \int_0^1 c(1-y) \left[\lim_{m \rightarrow \infty} p_m(x, y) \right] dy$$

In other words, we now have

$$p_\infty(x) = c(x) \star \left[\lim_{m \rightarrow \infty} P^m \right]$$

Putting these two expressions together yields the desired result.

⇐ Take

$$c(x) \star \left[\lim_{m \rightarrow \infty} P^m \right] = c(x) \star p_\infty(x) = p_\infty(x)$$

by definition of the limiting surface and **Claim 1.9**. The alternative way to write this by the stated equality in the hypothesis is:

$$\lim_{m \rightarrow \infty} [c(x) \star P^m] = p_\infty(x)$$

In turn, this can be written as $\lim_{m \rightarrow \infty} c_m(x) = p_\infty(x)$ by definition and we are done. \square

Corollary 4.5. (*January 15, 2013*)

$$\lim_{m \rightarrow \infty} \int_0^1 p_m(x, y) dy = p_\infty(x) = \lim_{m \rightarrow \infty} p_m(x, y)$$

Proof Take $c(x) = u(x)$, the uniform probability distribution, which equals 1 for all x in the domain. We can then state by **Corollary 4.4** that

$$\lim_{m \rightarrow \infty} [1 \star P^m] = 1 \star \left[\lim_{m \rightarrow \infty} P^m \right]$$

which is equivalent to:

$$\lim_{m \rightarrow \infty} \int_0^1 p_m(x, y) dy = \int_0^1 \lim_{m \rightarrow \infty} p_m(x, y) dy$$

which, in turn, yields

$$\lim_{m \rightarrow \infty} \int_0^1 p_m(x, y) dy = p_\infty(x) = \lim_{m \rightarrow \infty} p_m(x, y)$$

\square

Remark 4.1. (*January 15, 2013*) We now have that

$$\lim_{m \rightarrow \infty} \int_0^1 p_m(x, y) dx = 1$$

by **Definition 1.2** and **Claim 1.4 Closure of Pasquali Patches** and

$$\lim_{m \rightarrow \infty} \int_0^1 p_m(x, y) dy = p_\infty(x)$$

by **Corollary 4.5**.

Corollary 4.6. (*January 13, 2013*)

$$\lim_{m \rightarrow \infty} c_m(x) = p_\infty(x)$$

if and only if

$$\lim_{m \rightarrow \infty} [c_{m-1}(x) \star P^1] = \left[\lim_{m \rightarrow \infty} c_{m-1}(x) \right] \star P^1$$

Proof This follows from **Claim 4.2**. \square

Corollary 4.7. (*January 13, 2013*)

$$\lim_{m \rightarrow \infty} c_m(x) = p_\infty(x)$$

if and only if

$$c(x) \star \left[\lim_{m \rightarrow \infty} P^m \right] = \left[\lim_{m \rightarrow \infty} c_{m-1}(x) \right] \star P^1$$

In other words,

$$c(x) \star p_\infty(x) = c_\infty(x) \star P^1$$

Proof This follows from **Corollary 4.4** and **Corollary 4.6**. □

5. PASQUALI PATCH AND FUNCTION DYNAMICS

6. PROOFS IN PROGRESS

Generalizing the construction to $\sum f_i(x)g_i(y)$

Eigenvalues for surfaces

Groups defined on patchixes or patches

All states are achievable.

Alternative ways of writing the star operator (similarity to convolution).

The derivative as progressive shape pathix changes using converging \mathbb{P}

Continuous-time patchix changes

How are zeroes transferred

Claim 1.X. If a collection \mathbb{P} of *Pasquali patch* self-powers coincides with another \mathbb{Q} in at least one element, then either $\mathbb{P} \subset \mathbb{Q}$ or $\mathbb{Q} \subset \mathbb{P}$ or both.