COMPENDIUM OF CLAIMS AND PROOFS ON HOW PROBABILITY DISTRIBUTIONS TRANSFORM

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Contents

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1. The Star Product

Definition 1.1 (The Star Operator)**.** ***(October 17, 2010, January 17, 2013)***

- ***On Two Surfaces***

Let $f(x,y)$ and $g(x,y)$ be surfaces so that $f,g\colon [0,1]\times[0,1]\to\mathbb{R}$. The star operator $\star\colon [0,1]^2\times[0,1]^2\to[0,1]^2$ takes two surfaces and creates another in the following way:

$$(f(x,y),g(x,y)) \rightsquigarrow (f(1-y,z),g(x,y)) \rightsquigarrow h(x,z) \rightsquigarrow h(x,y)$$

with the central transformation being defined by $\diamond\colon [0,1]^2\times[0,1]^2\to[0,1]^2$

$$f(1-y,z)\diamond g(x,y) = \int_0^1 f(1-y,z)g(x,y)\,dy = h(x,z)$$

and the last transformation that takes $h(x,z)\rightsquigarrow h(x,y)$ we will call $j\colon [0,1]^2\to[0,1]^2$. Thus

$$\boxed{f(x,y)\star g(x,y) = j\left(f(1-y,z)\diamond g(x,y)\right) = j\left(\int_0^1 f(1-y,z)g(x,y)\,dy\right)}$$

- ***On a Function and a Surface***

Let $f(x)$ be a function $f\colon[0,1]\to\mathbb{R}$, and $g(x,y)$ a function such that $g\colon[0,1]^2\to\mathbb{R}$. The star operator $\star\colon[0,1]\times[0,1]^2\to[0,1]$ takes the function and the surface and creates another function in the following way:

$$(f(x),g(x,y)) \rightsquigarrow (f(1-y),g(x,y)) \rightsquigarrow h(x)$$

with the last transformation being defined by $\diamond\colon[0,1]\times[0,1]^2\to[0,1]$

$$f(1-y)\diamond g(x,y) = \int_0^1 f(1-y)g(x,y)\,dy = h(x)$$

Thus we have

$$\boxed{f(x)\star g(x,y) = f(1-y)\diamond g(x,y) = \int_0^1 f(1-y)g(x,y)\,dy}$$

1.1. Properties of the Diamond Operator.

1.1.1. *Linearity.*

Claim 1.1. ***(January 29, 2013)*** *The diamond operator is a linear operator.*

Proof Linearity follows from the integral operator properties. Thus we have:

- ***Surface diamond Surface***

Letting $f,g,h\colon[0,1]^2\to\mathbb{R}$, and c be a constant,

$$\begin{aligned}(c\cdot f)\diamond g &= \int_0^1 (c\cdot f(x,y))\,g(x,y)\,dy\\ &= c\int_0^1 f(x,y)g(x,y)\,dy\\ &= c\cdot(f\diamond g)\end{aligned}$$

$$\begin{aligned}f\diamond(c\cdot g) &= \int_0^1 f(x,y)\,(c\cdot g(x,y))\,dy\\ &= c\int_0^1 f(x,y)g(x,y)\,dy\\ &= c\cdot(f\diamond g)\end{aligned}$$

and

$$\begin{aligned}(f+g)\diamond h &= \int_0^1 (f(x,y)+g(x,y))\,h(x,y)\,dy\\ &= \int_0^1 f(x,y)h(x,y)\,dy + \int_0^1 g(x,y)h(x,y)\,dy\\ &= f\diamond h + g\diamond h\end{aligned}$$

$$\begin{aligned} f \diamond (g+h) &= \int_0^1 f(x,y)\left(g(x,y)+h(x,y)\right)\, dy \\ &= \int_0^1 f(x,y)g(x,y)\, dy + \int_0^1 f(x,y)h(x,y)\, dy \\ &= f \diamond g + f \diamond h \end{aligned}$$

- ***Function diamond Surface***

Letting $d, e\colon [0,1] \to \mathbb{R}$ and $f, g\colon [0,1]^2 \to \mathbb{R}$, and c be a constant,

$$\begin{aligned} (c \cdot d) \diamond f &= \int_0^1 \left(c \cdot d(x)\right) f(x,y)\, dy \\ &= c \int_0^1 d(x) f(x,y)\, dy \\ &= c \cdot (d \diamond f) \end{aligned}$$

$$\begin{aligned} d \diamond (c \cdot f) &= \int_0^1 d(x)\left(c \cdot f(x,y)\right)\, dy \\ &= c \int_0^1 d(x) f(x,y)\, dy \\ &= c \cdot (d \diamond f) \end{aligned}$$

and

$$\begin{aligned} (d+e) \diamond f &= \int_0^1 \left(d(x)+e(x)\right) f(x,y)\, dy \\ &= \int_0^1 d(x)f(x,y)\, dy + \int_0^1 e(x)f(x,y)\, dy \\ &= d \diamond f + e \diamond f \end{aligned}$$

$$\begin{aligned} d \diamond (f+g) &= \int_0^1 d(x)\left(f(x,y)+g(x,y)\right)\, dy \\ &= \int_0^1 d(x)f(x,y)\, dy + \int_0^1 d(x)g(x,y)\, dy \\ &= d \diamond f + d \diamond g \end{aligned}$$

□

1.2. Properties of the J Transformation.

1.2.1. *Linearity.*

Claim 1.2. ***(January 29, 2013)*** *The transformation $j\colon [0,1]^2 \to [0,1]^2$ is likewise linear.*

Proof Let $f, g\colon [0,1]^2 \to \mathbb{R}$, and c be a constant. Then:

$$\begin{aligned} j\left(c \cdot f(x,z)\right) &= c \cdot f(x,y) \\ &= c \cdot j\left(f(x,z)\right) \end{aligned}$$

Also:

$$\begin{aligned} j\left(f(x,z)+g(x,z)\right) &= f(x,y)+g(x,y) \\ &= j\left(f(x,z)\right) + j\left(g(x,z)\right) \end{aligned}$$

□

1.2.2. *Other Properties.*

Claim 1.3. ***(January 17, 2013)*** *Take the transformation $j\colon [0,1]^2 \to [0,1]^2$ that carries $h(x,z) \rightsquigarrow h(x,y)$. Then:*

$$\int_a^b j(h(x,z))\, dx = j\left(\int_a^b h(x,z)\, dx\right)$$

Proof

$$\int_a^b j(h(x,z))\,dx = \int_a^b h(x,y)\,dx = H(0,y)$$

On the other hand

$$j\left(\int_a^b h(x,z)\,dx\right) = j\left(H(0,z)\right) = H(0,y)$$

□

1.3. Properties of the Star Operator.

1.3.1. *Linearity.*

Corollary 1.4 (Scaling of the Star Product). ***(January 30, 2013)***
Let $d(x)$ be a function $d\colon [0,1] \to \mathbb{R}$, and $f(x,y)$ and $g(x,y)$ be functions $f,g\colon [0,1]^2 \to \mathbb{R}$, c is a constant. Then:

- **Surface star Surface**

$$(c\cdot f)\star g = c\cdot(f\star g)$$

and

$$f\star(c\cdot g) = c\cdot(f\star g)$$

- **Function star Surface**

$$(c\cdot d)\star f = c\cdot(d\star f)$$

and

$$d\star(c\cdot f) = c\cdot(d\star f)$$

Proof This is a consequence of **Claim 1.1** and **Claim 1.2**.

- ***Surface star Surface***

$$\begin{aligned}(c\cdot f)\star g &= j\left((c\cdot f(1-y,z))\diamond g(x,y)\right)\\ &= c\cdot j\left(f(1-y,z)\diamond g(x,y)\right)\\ &= c\cdot(f\star g)\end{aligned}$$

and

$$\begin{aligned}f\star(c\cdot g) &= j\left(f(1-y,z)\diamond(c\cdot g(x,y))\right)\\ &= c\cdot j\left(f(1-y,z)\diamond g(x,y)\right)\\ &= c\cdot(f\star g)\end{aligned}$$

- ***Function star Surface***

$$\begin{aligned}(c\cdot d)\star f &= j\left((c\cdot d(1-y))\diamond f(x,y)\right)\\ &= c\cdot j\left(d(1-y)\diamond f(x,y)\right)\\ &= c\cdot(d\star f)\end{aligned}$$

and

$$\begin{aligned}d\star(c\cdot f) &= j\left(d(1-y)\diamond(c\cdot f(x,y))\right)\\ &= c\cdot j\left(d(1-y)\diamond f(x,y)\right)\\ &= c\cdot(d\star f)\end{aligned}$$

□

Corollary 1.5 (Distributive Property of the Star Product). ***(January 27, 2013)***
Let $d(x), e(x)$ be a functions $d,e\colon [0,1]\to\mathbb{R}$ and $f(x,y), g(x,y)$ and $h(x,y)$ be functions $f,g,h\colon [0,1]^2\to\mathbb{R}$, c is a constant. Then:

- **Surface star Surface**

$$(f+g)\star h = f\star h + g\star h$$

and

$$f\star(g+h) = f\star g + f\star h$$

- **Function star Surface**

$$(d+e)\star f = d\star f + e\star f$$

and

$$d\star(f+g) = d\star f + d\star g$$

Proof Again this is a consequence of **Claim 1.1** and **Claim 1.2**.

- ***Surface star Surface***

We have:

$$\begin{aligned}(f+g)\star h &= j\left((f(1-y,z)+g(1-y,z))\diamond h(x,y)\right)\\ &= j\left(f(1-y,z)\diamond h(x,y)+g(1-y,z)\diamond h(x,y)\right)\\ &= j\left(f(1-y,z)\diamond h(x,y)\right)+j\left(g(1-y,z)\diamond h(x,y)\right)\\ &= f\star h+g\star h\end{aligned}$$

The second to last line is again a consequence of **Claim 1.2**.

The second part is:

$$\begin{aligned}f\star(g+h) &= j\left(f(1-y,z)\diamond(g(x,y)+h(x,y))\right)\\ &= j\left(f(1-y,z)\diamond g(x,y)+f(1-y,z)\diamond h(x,y)\right)\\ &= j\left(f(1-y,z)\diamond g(x,y)\right)+j\left(f(1-y,z)\diamond h(x,y)\right)\\ &= f\star g+f\star h\end{aligned}$$

where the second to last line is justified by **Claim 1.2**.

- ***Function star Surface***

Rewriting the first part,

$$\begin{aligned}(d+e)\star f &= (d(1-y)+e(1-y))\diamond f(x,y)\\ &= d(1-y)\diamond f(x,y)+e(1-y)\diamond f(x,y)\\ &= d\star f+e\star f\end{aligned}$$

follows directly from the linear properties of the diamond operator **Claim 1.1**.

Next, in the second part,

$$\begin{aligned}d\star(f+g) &= d(1-y)\diamond(f(x,y)+g(x,y))\\ &= d(1-y)\diamond f(x,y)+d(1-y)\diamond g(x,y)\\ &= d\star f+d\star g\end{aligned}$$

again by **Claim 1.1**. □

1.3.2. *Other Properties.*

Claim 1.6 (Zero-property)**.** ***(March 31, 2013)*** *Let f be a surface, with $f\colon[0,1]^2\to\mathbb{R}$. We have that*

$$0\star f = 0$$

The statement

$$f\star 0 = 0$$

is also true if $f\colon[0,1]\to\mathbb{R}$, f is a function.

Proof First,

$$\begin{aligned}0\star f &= \int_0^1 0\cdot f(x,y)\,dy\\ &= \int_0^1 0\,dy\\ &= 0\end{aligned}$$

Next, with $f\colon [0,1]^2 \to \mathbb{R}$,

$$\begin{aligned} f \star 0 &= \int_0^1 f(1-y,z)\cdot 0\,dy \\ &= \int_0^1 0\,dy \\ &= 0 \end{aligned}$$

and we can see that if $f\colon [0,1] \to \mathbb{R}$ we get

$$\begin{aligned} f \star 0 &= \int_0^1 f(1-y)\cdot 0\,dy \\ &= \int_0^1 0\,dy \\ &= 0 \end{aligned}$$

□

Claim 1.7 (Non-commutativity)**.** ***(October 17, 2010)*** *The star product is non-commutative.*

Proof by Counterexample The claim only makes sense from the viewpoint of surfaces.

- ***On Surfaces***

Let $f, g\colon [0,1]^2 \to \mathbb{R}$.

We want to show that $f(x,y) \star g(x,y) \neq g(x,y) \star f(x,y)$. Choose $f(x,y) = x$ and $g(x,y) = y$ without loss of generality. Then:

$$f(x,y) \star g(x,y) = f(x) \star g(x,y) = \int_0^1 f(1-y)\cdot g(x,y)\,dy = \int_0^1 (1-y)\cdot y\,dy = \int_0^1 \left(y - y^2\right)\,dy = \frac{1}{6}$$

and

$$g(x,y) \star f(x,y) = j\left(\int_0^1 g(1-y,z)\cdot f(x,y)\,dy\right) = j\left(\int_0^1 (z\cdot x)\,dy\right) = j\,(z\cdot x) = y\cdot x$$

are non-equal. □

Claim 1.8 (Associativity)**.** ***(October 17, 2010)*** *The star product is associative.*

Proof Again the claim only makes sense for the star operator on surfaces.

- ***On Surfaces***

Let $f, g, h\colon [0,1]^2 \to \mathbb{R}$.

We want to show that $[f(x,y) \star g(x,y)] \star h(x,y) = f(x,y) \star [g(x,y) \star h(x,y)]$.

The LHS is:

$$\begin{aligned} [f(x,y) \star g(x,y)] \star h(x,y) &= j\left(\int_0^1 f(1-y,z)g(x,y)\,dy\right) \star h(x,y) \\ &= j\left(\int_0^1 \int_0^1 f(1-z,w)g(1-y,z)\,dz\,h(x,y)\,dy\right) \\ &= j\left(\int_0^1 \int_0^1 f(1-z,w)g(1-y,z)\,h(x,y)\,dz\,dy\right) \end{aligned}$$

The RHS is:

$$\begin{aligned} f(x,y) \star [g(x,y) \star h(x,y)] &= f(x,y) \star j\left(\int_0^1 g(1-y,z)h(x,y)\,dy\right) \\ &= j\left(\int_0^1 f(1-z,w)\int_0^1 g(1-y,z)h(x,y)\,dy\,dz\right) \\ &= j\left(\int_0^1 \int_0^1 f(1-z,w)g(1-y,z)h(x,y)\,dy\,dz\right) \end{aligned}$$

We immediately see the equivalence using the Fubini Theorem to exchange the order of integration. □

1.4. Mechanics of Powering.

Claim 1.9 (Powering Symmetry)**.** ***(December 30, 2012)*** *Take $f\colon [0,1]^2 \to \mathbb{R}$. Let the nth power of $f(x,y)$, $f_n(x,y)$ be denoted shorthand by F^n (for* Pasquali patches[1]*, we use the notation P^n and $p_n(x,y)$ interchangeably). Then for $m, n \geq 1$, $F^n \star F^m = F^m \star F^n$.*

Proof by Induction By definition of powering,

$$F^1 \star F^m = F^1 \star \underbrace{(F^1 \star \ldots \star F^1)}_{\text{m times}}$$

Then, by **Claim 1.8 Associativity of the Star Product**, this is equivalent to

$$\underbrace{(F^1 \star \ldots \star F^1)}_{\text{m times}} \star F^1 = F^m \star F^1$$

Next assume that, for fixed $k, m \geq 1$, $F^k \star F^m = F^m \star F^k$.

Then $F^{k+1} \star F^m = (F^1 \star F^k) \star F^m$, where this bit we take as the definition of powering, and then, by **Claim 1.8** again, $F^1 \star (F^k \star F^m)$, which then by our inductive hypothesis equals $F^1 \star (F^m \star F^k)$ and this is $(F^m \star F^k) \star F^1$ by the inductive basis. Again, **Claim 1.8** gives $F^m \star (F^k \star F^1)$ and lastly $F^m \star F^{1+k}$ by definition of powering. Axiomatic commutativity of the positive integers gives $F^m \star F^{k+1}$ and we are done.

By symmetry of equality itself, we need only do a single proof of induction on one of the power parameters. □

Claim 1.10. ***(December 30, 2012)*** *Again take $f\colon [0,1]^2 \to \mathbb{R}$. $F^n \star F^m = F^{m+n}$.*

Proof by Induction First, fix $m \geq 1$. By the definition of powering, $F^1 \star F^m = F^{m+1}$. Next, suppose it's true that $F^k \star F^m = F^{m+k}$. Then $F^{k+1} \star F^m$ equals $(F^1 \star F^k) \star F^m$ by definition of powering, and by **Claim 1.8 Associativity of the Star Product** we get $F^1 \star (F^k \star F^m)$ or $F^1 \star F^{m+k}$ using the inductive hypothesis. By the inductive basis, this becomes $F^{(m+k)+1}$, that is $F^{m+(k+1)}$ using associativity of the positive integers, which we take as axiomatic.

Having done so, next fix $n \geq 1$ and let m vary. We have $F^n \star F^1 = F^1 \star F^n$ by **Claim 1.9 Powering Symmetry**. Since this part of the proof is identical to the one we just wrote, we are done. □

1.4.1. *Special Subcollections of Powers.*

Remark 1.1. ***(January 13, 2014)*** *If we accept powering of functions via the defintions of the star product, then the following will make sense. On the other hand if we view this from the standpoint of simple matrices (and replace the star product by matrix multiplication), the following claims may also make sense.*

Definition 1.2. ***(January 13, 2014)*** *For a countably infinite collection $\mathbb{F} = \{F^1, F^2, \ldots, F^j, \ldots\}_{j \in \mathbb{Z}^+}$, the* generator *is F^1 since all other elements in the collection are powers of such. (As an aside, we may contemplate finite collections in the event of powering periodicity or nilpotency, but not presently).*

Claim 1.11. ***(January 13, 2014)*** *Take the countably infinite collections $\mathbb{F}$ and $\mathbb{G}$ with generators F^1 and G^1 respectively. If $F^1 = G^1$, then $\mathbb{F} = \mathbb{G}$.*

Proof by Induction By hypothesis we have that $F^1 = G^1$, and let us assume that $F^k = G^k$. Then:

$$F^{k+1} = F^1 \star F^k = G^1 \star G^k = G^{k+1}$$

Thus we have shown that $\mathbb{F} \subset \mathbb{G}$ and $\mathbb{G} \subset \mathbb{F}$ and so naturally we conclude $\mathbb{F} = \mathbb{G}$. □

Claim 1.12. ***(January 13, 2014)*** *Take the countably infinite collections $\mathbb{F}$ and $\mathbb{G}$ with generators F^1 and G^1 respectively. If $F^1 = G^n$ for some $n \in \mathbb{Z}^+$, then $\mathbb{F} \subset \mathbb{G}$.*

Proof by Induction By hypothesis $F^1 = G^n$ for some positive integer n. Also,

$$F^2 = F^1 \star F^1 = G^n \star G^n = G^{2n}$$

using **Claim 1.10**. Next suppose that $F^k = G^{kn}$. Then:

$$F^{k+1} = F^1 \star F^k = G^n \star G^{kn} = G^{n+kn} = G^{(k+1)n}$$

Thus $\mathbb{F} \subset \mathbb{G}$ since all elements of the collection $\mathbb{F}$ are contained in $\mathbb{G}$. □

[1] See **Section 2**

Definition 1.3 (Subsequent Power Subset at n)**.** ***(January 13, 2014)*** *For a countably infinite collection* $\mathbb{F} = \{F^1, F^2, \ldots, F^j, \ldots\}_{j\in\mathbb{Z}^+}$*, we call the subset* $\mathbb{F}_n \subset \mathbb{F}$ *the* subsequent power subset at n*, and it is composed of all those powers greater than or equal to* $n \in \mathbb{Z}^+$*.*

Claim 1.13. ***(January 13, 2014)*** *Take the countably infinite collections* $\mathbb{F}$ *and* $\mathbb{G}$ *with generators* F^1 *and* G^1 *respectively. If* $F^m = G^n$ *for some* $m, n \in \mathbb{Z}^+$ *and* $m < n$ *(since we haven't defined* roots *it must also be true that* $\frac{n}{m} \in \mathbb{Z}^+$*), then* $\mathbb{F}_m \subset \mathbb{G}_n \subset \mathbb{G}$*. (At this point we don't really care about containment of powers less than* n*, since we haven't established a mechanism to obtain them by* unpowering *or* taking roots*).*

Proof by Induction By hypothesis $F^m = G^n$. In particular, we have the statement $F^{m+1} = G^{n+p}$ for some integer $p \in \mathbb{Z}^+$. (Advancing one element in the collection $\mathbb{F}$ starting from m, advances us p units in the collection $\mathbb{G}$ starting from n.) Taking this statement to the logical consequence, $F^m \star F^1 = G^n \star G^p \Rightarrow F^1 = G^p$. (Notice that since we have not defined negative powers of functions, p is necessarily a positive integer).

Next let us assume that $F^{m+k} = G^{n+kp}$. Then:

$$F^{m+(k+1)} = F^{(m+k)+1} = F^{m+k} \star F^1 = G^{n+kp} \star G^p = G^{n+(k+1)p}$$

It can be seen now that $\mathbb{F}_m \subset \mathbb{G}_n \subset \mathbb{G}$. □

Corollary 1.14. ***(January 13, 2014)*** *Take the countably infinite collections* $\mathbb{F}$ *and* $\mathbb{G}$*.* $\mathbb{F}_m = \mathbb{G}_n \subset \mathbb{G}$ *if and only if* $m = n$ *and* $p = 1$*.*

Proof We have:

- $\Rightarrow$ Since $\mathbb{F}_m = \mathbb{G}_n$ there is no element of either collection that does not match one in the other collection. Thus the step between elements must be 1, and $p = 1$. Furthermore the elements in a collection can be put in bijective correspondence (by definition of equality of sets) with the elements in the other collection, and the indices must exactly coincide by the ordering being the usual order on the positive integers (we order by power). Thus $m = n$.
- $\Leftarrow$ **(Proof by Induction)** Since $m = n$ we have that $F^m = G^n \Rightarrow F^m = G^m$ for all $m \in \mathbb{Z}^+$, and also $F^{m+1} = G^{n+p} \Rightarrow F^{m+1} = G^{m+1}$ since $p = 1$. So now assume that this works for the kth element $k > m+1$, so that $F^{m+k} = G^{m+k} \Rightarrow F^k = G^k$. Then we have that:

$$F^{m+(k+1)} = F^{(m+1)+k} = F^{(m+1)} \star F^k = G^{(m+1)} \star G^k = G^{(m+1)+k} = G^{m+(k+1)}$$

and we are done.

□

2. *Pasquali Patches*

2.1. **Definitions and Constructions.**

2.1.1. *Main Definition.*

Definition 2.1 (*Pasquali patch*)**.** ***(April 22, 2010)*** *Define a continuous, bounded surface* $p(x, y)$*, with* $p\colon [0,1] \times [0,1] \to \mathbb{R}^+ \cup \{0\}$*, and let* $\int_0^1 p(x,y)\,dx = 1$ *be true regardless of the value of* y*. In other words, integrating such surface with respect to* x *yields the uniform probability distribution* $u(y)$*,* $u\colon [0,1] \to \{1\}$*. We will call this a* strict Pasquali patch*, and such is intimately related to probability notions. With* $p\colon [0,1] \times [0,1] \to \mathbb{R}$*, we have a more general definition for a* Pasquali patch*.*

Claim 2.1. ***(March 20, 2016)*** *The function* $p(x,y) = p(x) = \frac{f(x)}{F}$*, where* $F = \int_0^1 f(x)\,dx \neq 0$*, is a* Pasquali patch*.*

Proof Using **Definition 2.1**, let us integrate across x:

$$\int_0^1 p(x,y)\,dx = \int_0^1 p(x)\,dx = \int_0^1 \frac{f(x)}{F}\,dx = \frac{F}{F} = u(y) = 1$$

□

Claim 2.2. ***(March 18, 2016)*** *A* Pasquali patch *cannot consist of a function of* x *times an explicit function of* y*. In other words, its form cannot be* $p(x,y) = f(x) \cdot g(y)$*.*

Proof Suppose that the form of the *Pasquali patch* is $p(x,y) = f(x)\cdot g(y)$. By **Definition 2.1**, $\int_0^1 p(x,y)\,dx = 1$. Take

$$\begin{aligned} \textstyle\int_0^1 p(x,y)\,dx &= \textstyle\int_0^1 f(x)\cdot g(y)\,dx \\ &= g(y)\textstyle\int_0^1 f(x)\,dx \\ &= g(y)\cdot F \end{aligned}$$

is a function of y, not a constant and in particular not a constant equal to 1. □

2.1.2. *Main Construction.*

Construction 2.1. ***(October 10, 2010)*** *A way to construct a* Pasquali patch *is by positing*

$$p(x,y) = f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)\int_0^1 f_1(x)\,dx}{\int_0^1 f_2(x)\,dx} = f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)F_1}{F_2}$$

for arbitrary function choices $f_1(x)$ and $g_1(y)$, and $f_2(x)$ is chosen so that its integral $F_2 \neq 0$.

Proof Since for *Pasquali patches* $\int_0^1 p(x,y)\,dx = 1$, it can be seen that such construction is a *Pasquali patch* because

$$\int_0^1 \left(f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)F_1}{F_2} \right) dx = g_1(y)F_1 + \frac{F_2}{F_2} - \frac{g_1(y)F_1F_2}{F_2} = 1$$

□

Claim 2.3. ***(March 20, 2016)*** *We can redefine the function $f_2(x)$ as $f_2^*(x)$ so that it is always a* Pasquali patch, *and write*

$$p(x,y) = f_1(x)g_1(y) - f_2^*(x)g_1(y)F_1 + f_2^*(x) = (f_1(x) - f_2^*(x)\cdot F_1)\cdot g_1(y) + f_2^*(x)$$

with $F_2^* = 1$. Observe the unit contribution to the integral of the Pasquali patch *is exactly provided by $f_2^*(x)$.*

Proof We had almost complete freedom when choosing the function $f_2(x)$, in that the only restriction was that $F_2 \neq 0$. We can define the normalized function $f_2^*(x) = \frac{f_2(x)}{F_2}$, $F_2 \neq 0$. Then, by **Claim 2.1**, $f_2^*(x)$ is a *Pasquali patch* as we wanted to show, with $F_2^* = 1$. We may now recast **Construction 2.1** in its normalized form by substituting into the original formulation appropriately. □

Generalization of **Construction 2.1**.

Construction 2.2. ***(March 18, 2016)*** *Let $p(x,y) = \sum_{i=1}^n f_i(x)\cdot g_i(y) = \mathbf{f}(x)\cdot\mathbf{g}(y)$, a function which consists of a finite sum of pairs of functions of x and y.*[2] *Then $p(x,y)$ is a* Pasquali patch *provided*

$$g_n(y) = \frac{1 - \sum_{i=1}^{n-1} g_i(y)F_i}{F_n} = \frac{1 - \mathbf{g}_{n-1}(y)\cdot\mathbf{F}_{n-1}}{F_n}$$

and $F_n \neq 0$. Thus, we may choose $n-1$ arbitrary functions of x, $n-1$ arbitrary functions of y, an nth function of x so that $F_n \neq 0$, and

$$p(x,y) = \sum_{i=1}^{n-1} f_i(x)\cdot g_i(y) + f_n(x)\cdot\frac{1 - \sum_{i=1}^{n-1} g_i(y)F_i}{F_n} = \mathbf{f}_{n-1}(x)\cdot\mathbf{g}_{n-1}(y) + f_n(x)\cdot\frac{1 - \mathbf{g}_{n-1}(y)\cdot\mathbf{F}_{n-1}}{F_n}$$

We may write the normalized version as:

$$\boxed{p(x,y) = (\mathbf{f}_{n-1}(x) - f_n^*(x)\cdot\mathbf{F}_{n-1})\cdot\mathbf{g}_{n-1}(y) + f_n^*(x)}$$

and again observe that the unit contribution to the integral of the Pasquali patch *is provided by $f_n^*(x)$, so that $F_n^* = 1$.*

Proof We use the normalized definition which gives the quicker proof.

$$\begin{aligned} \textstyle\int_0^1 p(x,y)\,dx &= \textstyle\int_0^1 ((\mathbf{f}_{n-1}(x) - f_n^*(x)\cdot\mathbf{F}_{n-1})\cdot\mathbf{g}_{n-1}(y) + f_n^*(x))\,dx \\ &= \mathbf{g}_{n-1}(y)\cdot\textstyle\int_0^1 (\mathbf{f}_{n-1}(x) - f_n^*(x)\cdot\mathbf{F}_{n-1})\,dx + \int_0^1 f_n^*(x)\,dx \\ &= \mathbf{g}_{n-1}(y)\cdot(\mathbf{F}_{n-1} - F_n^*\cdot\mathbf{F_{n-1}}) + 1 \\ &= 1 \end{aligned}$$

We may now write $p(x,y)$ is a *Pasquali patch* as we wanted to show. □

[2]In the spirit of conciseness, we omit the transpose symbology, thusly understanding the first vector listed in the dot product as a row vector and the second vector as a column vector.

2.1.3. *Secondary Construction.*

Construction 2.3. ***(March 9, 2016)*** *A simplification of* **Construction 2.1** *arises when we consider* $\int_0^1 f_1(x)\,dx = F_1 = 0$*, yielding* $p(x,y) = f_1(x)g_1(y) + f_2^*(x)$*. This simplification is still a* Pasquali patch.

Proof We need only consider the normalized version of **Construction 2.1** to obtain the desired result:

$$p(x,y) = \left(f_1(x) - \cancel{f_2^*(x) \cdot F_1}\right) \cdot g_1(y) + f_2^*(x) = f_1(x) \cdot g_1(y) + f_2^*(x)$$

Since this is a restriction of **Construction 2.1**, it must also be a *Pasquali patch*. We can show this quickly via integration, however:

$$\int_0^1 p(x,y)\,dx = \int_0^1 f_1(x)g_1(y)\,dx + \int_0^1 f_2^*(x)\,dx = \cancel{g_1(y)F_1} + 1 = 1$$

□

Generalization of **Construction 2.3**.

Construction 2.4. ***(March 18, 2016)*** *The function*

$$\boxed{p(x,y) = \sum_{i=1}^{n-1} f_i(x) \cdot g_i(y) + \frac{f_n(x)}{F_n} = \mathbf{f}_{n-1}(x) \cdot \mathbf{g}_{n-1}(y) + f_n^*(x)}$$

is a Pasquali patch *provided* $F_i = 0, i \in \{1, \ldots, n-1\}$*,* $F_n^* = 1$.

Proof Using the normalized version in **Construction 2.2**, we have:

$$p(x,y) = (\mathbf{f}_{n-1}(x) - \cancel{f_n^*(x) \cdot \mathbf{F}_{n-1}}) \cdot \mathbf{g}_{n-1}(y) + f_n^*(x) = \mathbf{f}_{n-1}(x) \cdot \mathbf{g}_{n-1}(x) + f_n^*(x)$$

and we are done because the construction is a restriction of **Construction 2.2**. We may check the integral for good measure:

$$\int_0^1 p(x,y)\,dx = \int_0^1 \mathbf{f}_{n-1}(x) \cdot \mathbf{g}_{n-1}(y)\,dx + \int_0^1 f_n^*(x)\,dx = \cancel{\mathbf{g}_{n-1}(y) \cdot \mathbf{F}_{n-1}} + 1 = 1$$

□

2.2. Properties of *Pasquali Patches* Under the Star Operator.

2.2.1. *Remarkable Properties.*

Claim 2.4 (Closure of *Pasquali Patches*)**.** ***(October 12, 2010)*** *A* Pasquali patch *star a* Pasquali patch *yields a new* Pasquali patch*. In particular, a* Pasquali patch *star itself (*Pasquali patch *powers) will always yield another* Pasquali patch.

Proof Take *Pasquali patches* $p(x,y)$ and $q(x,y)$. We want to show that $p(x,y) \star q(x,y) = r(x,y)$ is a *Pasquali patch.*

Thus, we want to show that $\int_0^1 r(x,y)\,dx = 1$. In other words,

$$\int_0^1 j\left(\int_0^1 p(1-y,z)q(x,y)\,dy\right)\,dx = 1$$

By **Claim 1.3**, we can write this as

$$j\left(\int_0^1 \int_0^1 p(1-y,z)q(x,y)\,dy\,dx\right)$$

We can exchange the order of integration because of absolute convergence of the integrals (Fubini Theorem). Thus

$$j\left(\int_0^1 p(1-y,z)\int_0^1 q(x,y)\,dx\,dy\right)$$

yields

$$j\left(\int_0^1 p(1-y,z)u(y)\,dy\right) = j\left(\int_0^1 p(1-y,z)\,dy\right)$$

This final integral evaluates to $j\,(u(z)) = u(y) = 1$ for any choice of y. Thus $\int_0^1 r(x,y)\,dx = 1$. □

Example 2.1. ***(October 12, 2010)*** *Take the* Pasquali patch $p(x,y) = x^2y^3 + x\left(2 - \frac{2y^3}{3}\right)$*. It is evident it is a* Pasquali patch *because*

$$\int_0^1 \left[x^2y^3 + x\left(2 - \frac{2y^3}{3}\right)\right] dx = \left[\frac{x^3}{3}y^3 + \frac{x^2}{2}\left(2 - \frac{2y^3}{3}\right)\right]\Bigg|_0^1 = 1$$

We can calculate

$$p(1-y,z) = z^3y^2 - \frac{4z^3y}{3} + \frac{z^3}{3} - 2y + 2$$

so that the second "power" of the Pasquali patch *is*

$$p_2(x,y) = j\left(\int_0^1 p(1-y,z)p(x,y)\,dy\right) = j\left(\frac{29x}{15} + \frac{z^3x}{90} + \frac{x^2}{10} - \frac{z^3x^2}{60}\right)$$

and the final transformation j of the star operator gives

$$p_2(x,y) = \frac{29x}{15} + \frac{y^3x}{90} + \frac{x^2}{10} - \frac{y^3x^2}{60}$$

One easily checks $p_2(x,y)$ is indeed a Pasquali patch *by performing the integral $\int_0^1 p_2(x,y)\,dx$ and ascertaining its equality to* 1*. The third* Pasquali patch *power is*

$$p_3(x,y) = \frac{1741x}{900} - \frac{y^3x}{5400} + \frac{59x^2}{600} + \frac{y^3x^2}{3600}$$

Example 2.2. ***(October 15, 2010, October 17, 2010)*** *The* Pasquali patch

$$p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$$

has powers:

$$\begin{aligned}
p(x,y) &= 1 - \cos(2\pi x)\cos(2\pi y)\\
p_2(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2}\\
p_3(x,y) &= 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{4}\\
p_4(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{8}\\
p_5(x,y) &= 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{16}\\
p_6(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{32}\\
&\vdots\\
p_n(x,y) &= 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}
\end{aligned}$$

Proof by Induction We show that, by the inductive basis, $p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$ must equal

$$p_1(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(0)}}$$

which a quick check shows is indeed the case.

Next, by the inductive hypothesis, we take as true that:

$$p_k(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(k-1)}}$$

Then,

$$\begin{aligned}
p_{k+1}(x,y) &= j\left(\int_0^1 p_1(1-y,z)\cdot p_k(x,y)\,dy\right)\\
&= j\left(\int_0^1 (1 - \cos(2\pi(1-y))\cos(2\pi z))\cdot\left(1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{k-1}}\right)dy\right)
\end{aligned}$$

The product of 1 with itself is 1, and such will integrate to 1 in the unit interval. So we save it. The integrals $\int_0^1 \cos(2\pi y)\,dy$ and $\int_0^1 \cos(2\pi - 2\pi y)\,dy$ both evaluate to zero, so we are left only with the task of evaluating the crossterm:

$$\begin{aligned}\int_0^1 \cos(2\pi(1-y))\cos(2\pi z)\cdot\frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{k-1}}\,dy &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}}\int_0^1 \cos(2\pi - 2\pi y)\cos(2\pi y)\,dy \\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}}\int_0^1 \cos^2(2\pi y)\,dy \\ &= \frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^{k-1}}\cdot\frac{1}{2} \\ &= -\frac{\cos(2\pi z)\cos(2\pi x)}{(-2)^k}\end{aligned}$$

Let's not forget the 1 we had saved, so:

$$p_{k+1}(x,y) = j\left(1 - \frac{\cos(2\pi x)\cos(2\pi z)}{(-2)^k}\right) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^k}$$

as we wanted to show. □

Remark 2.1. ***(April 22, 2010) Probabilistic Interpretation.*** *In direct analogy to the Chapman-Kolmogorov equation, transition from state $y \in [0,1]$ to $x \in [0,1]$ in $n+m$ steps can be achieved by first transitioning to an intermediate state $x^\bullet$ in n steps and then jumping from there to x in m more steps. Particularly, all states are achievable from a starter state since for* Pasquali patch $p(x,y)$ *there are only a finite number of zeroes in the domain because the surface is well-behaved.*

2.2.2. *Other Properties.*

Claim 2.5 (Non-commutativity of *Pasquali Patches* Under the Star Product)**.** ***(October 17, 2010)*** *We know in general the star product is non-commutative (**Claim 1.7**), but we don't know that* Pasquali patches *as a subset of surfaces with domain on* $[0,1]\times[0,1]$ *("*Pasquali patchixes*", due to their resembling matrixes in that they transform functions to other functions, as matrixes transform vectors to other vectors) don't commute.*

Proof by Counterexample Suppose the *Pasquali patches* $p(x,y) = x + \frac{1}{2}$ and $q(x,y) = 1 + xy - \frac{y}{2}$. Then

$$\begin{aligned}p(x,y)\star q(x,y) = p(x)\star q(x,y) &= \int_0^1 p(1-y)\cdot q(x,y)\,dy \\ &= \int_0^1 \left(\frac{3}{2} - y\right)\cdot\left(1 + xy - \frac{y}{2}\right)\,dy \\ &= \frac{5x}{12} + \frac{19}{24}\end{aligned}$$

where

$$\begin{aligned}q(x,y)\star p(x,y) &= j\left(\int_0^1 q(1-y,z)\cdot p(x,y)\,dy\right) \\ &= j\left(\int_0^1 q(1-y,z)\cdot p(x)\,dy\right) \\ &= j\left(p(x)\int_0^1 q(1-y,z)\,dy\right) \\ &= j\left(p(x)\cdot u(z)\right) = p(x) \\ &= x + \frac{1}{2}\end{aligned}$$

□

Corollary 2.6 (Associativity of *Pasquali Patches* Under the Star Product)**.** ***(October 17, 2010, December 3, 2012)*** Pasquali patches *inherit associativity.*

Proof *Pasquali patches* inherit associativity from the star product through **Claim 1.8 Associativity of the Star Product**. □

2.3. **Limiting Surface.**

Definition 2.2. *The limiting, steady-state, or stationary surface is a surface obtained by taking the limit as n approaches infinity of* Pasquali patch *powers* $\lim_{n\to\infty} p_n(x,y)$*, provided it exists, and we call it* $p_\infty(x,y)$.

Lemma 2.7. ***(October 17, 2010)*** *The stationary surface of*

$$p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$$

is $p_\infty(x,y) = 1$.

Proof Since the *Pasquali patch* power collection of $p(x,y) = 1 - \cos(2\pi x)\cos(2\pi y)$ can be described by

$$p_n(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}$$

due to **Example 2.2** we need only take the limit as n approaches infinity of this formula:

$$\lim_{n\to\infty} p_n(x,y) = \lim_{n\to\infty}\left(1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}\right) = 1$$

In this particular case, notice that $p_\infty(x,y) = 1$ is also a *Pasquali patch* because

$$\int_0^1 p_\infty(x,y)\,dx = 1$$

□

Lemma 2.8. ***(December 3, 2012)*** *If the sequence* $p_n(x,y)$ *of* Pasquali patch *powers converge uniformly to* $p_\infty(x,y)$*, then* $p_\infty(x,y)$ *is a* Pasquali patch.

Proof We want to show that $\int_0^1 p_\infty(x,y)\,dx = 1$. First,

$$\int_0^1 p_\infty(x,y)\,dx = \int_0^1 \lim_{n\to\infty} p_n(x,y)\,dx$$

By uniform convergence of the sequence of *Pasquali patches*, we can exchange the order of the limit to obtain

$$\lim_{n\to\infty}\int_0^1 p_n(x,y)\,dx$$

Now for any n, $p_n(x,y)$ is a Pasquali patch, and therefore the integral is equal to 1 in every case (for any choice of n). Lastly,

$$\lim_{n\to\infty}\{1\}_n = 1$$

and we have arrived at what we wanted to show. □

Claim 2.9. ***(December 3, 2012)*** Pasquali patches *that are functions of y (possibly x) explicitly generate by self-powering a countably infinite collection of* Pasquali patches *that are functions of y (possibly x) explicitly.*

Proof by Induction Let $p(x,y)$ be a *Pasquali patch* that is a function of y (possibly x) explicitly. Thus the basis is true by hypothesis. By the inductive hypothesis, let $p_k(x,y)$, $k \in \mathbb{Z}^+$ be an explicit function of y (possibly x). Then:

$$p_{k+1}(x,y) = p_k(x,y) \star p(x,y) = j\left(\int_0^1 p_k(1-y,z)p(x,y)\,dy\right)$$

is an explicit function of y (possibly x) since $p_k(x,y)$ was an explicit function of y (possibly x), and the y was saved by the transformation to z in $p_k(1-y,z)$, so that the integral did not aggregate it. Then, the transformation j takes $z \rightsquigarrow y$ makes $p_{k+1}(x,y)$ an explicit function of y (possibly x). The result is a countable collection of explicit functions of y (possibly x) via self-powering, because $k \in \mathbb{Z}^+$. □

Claim 2.10. ***(October 17, 2010)*** *A* Pasquali patch *star a* Pasquali patch *that is solely a function of x returns the second* Pasquali patch.

Proof Suppose the *Pasquali patches* $q(x,y)$ and $p(x,y) = p(x)$. Then:

$$\begin{aligned} q(x,y) \star p(x,y) &= j\left(\int_0^1 q(1-y,z) \cdot p(x)\,dy\right) \\ &= j\left(p(x)\int_0^1 q(1-y,z)\,dy\right) \\ &= j\left(p(x) \cdot u(z)\right) = p(x) \end{aligned}$$

□

Claim 2.11. ***(December 3, 2012)*** Pasquali patches *that are functions of x alone (explicitly or otherwise) generate* Pasquali patches *that are the same as the original* Pasquali patch *via self-powering.*

Proof by Induction $p(x) = p(x)$ by the identity of equality. $p_2(x) = p(x) \star p(x) = p(x)$ by **Claim 2.10**. This establishes the induction basis. Next, suppose $p_k(x) = p(x)$. Then $p_{k+1}(x) = p_k(x) \star p(x) = p(x) \star p(x) = p(x)$, and we are done. □

Claim 2.12. ***(December 3, 2012)*** *If the limiting surface $p_\infty(x,y)$ exists, it is NOT an explicit function of y. It is either an explicit function of x or constant for all x, y. We can describe it WLOG by $p_\infty(x)$.*

Proof The proof consists of two cases.

Case 1. Suppose we have a generator *Pasquali patch* $p(x)$ which is a function of x alone (explicitly or otherwise). Then, by **Claim 2.11**, $\forall n \in \mathbb{Z}^+$, $p_n(x) = p(x)$. Taking the limit as n approaches infinity, we obtain $\lim_{n\to\infty} p_n(x) = \lim_{n\to\infty} p(x)$, or $p_\infty(x) = p(x)$. This limiting surface is therefore also a *Pasquali patch*.

Case 2. In the space of uniformly convergent surfaces generated by *Pasquali patch* self-powers, the limiting surface will exist and will be a *Pasquali patch* itself by **Lemma 2.8**. Now suppose that $p_\infty(x,y)$ is an explicit function of y (possibly x). Then either it belongs to the collection of *Pasquali patch* powers generated by $p(x,y)$, or to a different collection altogether generated by *Pasquali patch* powers of, say, $q(x,y)$. If it belongs to the collection of *Pasquali patch* powers generated by $p(x,y)$, it must be equal to $p_n(x,y)$ for some n. But such is not an accumulation surface because we can generate $p_{n+1}(x,y)$ and onwards. It therefore must belong to a different collection, that generated by $q(x,y)$, and equals $q_m(x,y)$ for some m. The problem is that such isn't an accumulation surface either, being part of a (different) sequence of *Pasquali patch* powers, and in a "parallel" collection. We are faced with a contradiction. $p_\infty(x,y)$ must therefore NOT be an explicit function of y: it is either explicitly or not a function of x, that is, $p_\infty(x)$. Recall it is also a *Pasquali patch* by **Lemma 2.8**.

□

2.4. Probability Distribution Transformations via *Pasquali Patches*.

Claim 2.13. ***(October 10, 2010)*** *A well-behaved continuous probability distribution $b\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 b(x)\,dx = 1$,* ***star*** *a* Pasquali patch*, yields a continuous probability distribution $c\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 c(x)\,dx = 1$. In other words, a probability distribution $b(x)$ is taken to a probability distribution $c(x)$ via the* Pasquali patch*: $b(x) \rightsquigarrow c(x)$.*

Proof 1 Let $c(x) = b(x) \star p(x,y)$. We seek to show that

$$\int_0^1 c(x)\,dx = \int_0^1 b(x) \star p(x,y)\,dx = 1$$

Using **Definition 1.1**,

$$\int_0^1 b(x) \star p(x,y)\,dx = \int_0^1 \int_0^1 b(1-y)p(x,y)\,dy\,dx$$

Absolute convergence of the integrals allows us to exchange the order of integration (Fubini Theorem). Thus:

$$\int_0^1 \int_0^1 b(1-y)p(x,y)\,dx\,dy = \int_0^1 b(1-y) \int_0^1 p(x,y)\,dx\,dy$$

The innermost integral adds up to $u(y) = 1$ by **Definition 2.1**. Next

$$\int_0^1 b(1-y)u(y)\,dy = \int_0^1 b(1-y) \cdot 1\,dy = 1$$

by virtue of $b(x)$ being a probability distribution. □

Proof 2 Via **Closure of *Pasquali Patches* (Claim 2.4)**, a continuous probability distribution $b(x)$ can be thought of as $b(x, y)$ and a *Pasquali patch*, thus it **star** another *Pasquali patch* will yield a new *Pasquali patch* by closure of *Pasquali patches*. □

Example 2.3. ***(October 10, 2010)*** *Let $b(x) = 6x(1 - x)$. This is a beta(2,2) probability distribution and*

$$\int_0^1 (6x(1-x))\,dx = \int_0^1 \left(6x - 6x^2\right)\,dx = \left.(3x^2 - 2x^3)\right|_0^1 = 1$$

is easily checked. Let $p(x, y) = x + \frac{1}{2}$ be a Pasquali patch, *with*

$$\int_0^1 \left(x + \frac{1}{2}\right)\,dx = \left.\left(\frac{x^2}{2} + \frac{x}{2}\right)\right|_0^1 = 1$$

for any choice of y. Then

$$b(x) \star p(x, y) = \int_0^1 b(1-y)p(x, y)\,dy = \int_0^1 6(1-y)(y)\left(x + \frac{1}{2}\right)\,dy = x + \frac{1}{2}$$

This probability distribution has already been shown to integrate to 1 with respect to x.

Corollary 2.14. ***(October 10, 2010)*** *A continuous probability distribution $b\colon [0, 1] \to \mathbb{R}^+ \cup \{0\}$* ***star*** *a* Pasquali patch *that solely a function of x yields a continuous probability distribution that is of the same form as the* Pasquali patch. *In other words, a probability distribution $b(x)$ is carried via the* Pasquali patch *$p(x, y) = p(x)$ to $p(x)$: $b(x) \rightsquigarrow p(x)$.*

Proof 1 Using the definition of the star operator, $\int_0^1 b(1-y)p(x, y)\,dy = \int_0^1 b(1-y)p(x)\,dy = p(x)$. □

Proof 2 Via **Claim 2.10**. A continuous probability distribution can be thought of as a *Pasquali patch*, and thus **Claim 2.10** applies. □

2.5. Fixed Distribution Conjecture.

Conjecture 2.1. ***(December 3, 2012)*** *It is only natural to ask if there exists a (bounded, well-behaved, probability) function $a(x)$, $a\colon [0, 1] \to \mathbb{R}^+ \cup \{0\}$ that remains fixed when starred by a* Pasquali patch *$p(x, y)$. In other words, is there an $a(x)$ so that $a(x) \star p(x, y) = a(x)$?*

Claim 2.15. ***(December 3, 2012)*** *Suppose* ***Conjecture 2.1*** *is true. Then $a(x)$ remains fixed for all* Pasquali patch *powers generated by $p(x, y)$.*

Proof by Induction We take as true that $a(x) \star p(x, y) = a(x)$ by **Conjecture 2.1**. By the inductive hypothesis, we have that $a(x) \star p_k(x, y) = a(x)$. Then:

$$a(x) \star p_{k+1}(x, y) = a(x) \star (p_k(x, y) \star p(x, y))$$

By **Claim 1.8 Associativity of the Star Product**, this last expression we can write as:

$$(a(x) \star p_k(x, y)) \star p(x, y) = a(x) \star p(x, y) = a(x)$$

and we are done. □

Corollary 2.16. ***(July 23, 2013)*** *Consider the more general eigenvalue problem*

$$a(x) \star p(x, y) = \lambda\, a(x)$$

with $p(x, y)$ is a Pasquali patch. *Take* ***Conjecture 2.1*** *is true. Then $\lambda = 1$ is an eigenvalue of all self-powers of $p(x, y)$.*

Proof The proof follows from **Claim 2.15**, since $a(x)$ remains exactly fixed for all *Pasquali patch* powers generated by $p(x, y)$, implying $\lambda = 1$. □

Claim 2.17. ***(December 3, 2012)*** *The only function $b(x)$ that makes true the expression $a(x) \star b(x) = a(x)$, $a(x)$ is a probability distribution $a\colon [0, 1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 a(x)\,dx = 1$ (and therefore a* Pasquali patch *from another viewpoint), is $a(x)$ itself.*

Proof 1 We resort to the definition of the star operator, and the LHS is

$$a(x) \star b(x) = \int_0^1 a(1-y)b(x)\,dy = b(x)\int_0^1 a(1-y)\,dy = b(x)$$

On the other hand, the RHS is $a(x)$, and $b(x) = a(x)$. □

Proof 2 By **Claim 2.11**, the LHS is $b(x)$. The RHS is $a(x)$ and the equality is established. □

Corollary 2.18. ***(December 3, 2012)*** $a(x) \star a(x) = a(x)$

Proof By **Claim 2.17**, since the only function of x that makes the expression true is $a(x)$, the result follows by direct plugging-in to the original expression. □

Corollary 2.19. ***(December 3, 2012)*** *$a(x)$ is fixed for the limiting* Pasquali patch *probability distribution of some collection of uniformly convergent* Pasquali patch *self-powers of $p(x, y)$ if and only if $p_\infty(x) = a(x)$.*

Proof We have:

- $\Rightarrow$ By **Claim 2.12**, the *Pasquali patch* self-powers of $p(x, y)$ converge to $p_\infty(x)$, a *Pasquali patch* (by **Lemma 2.8**) which is NOT explicitly a function of y. By hypothesis, $a(x) \star p_\infty(x) = a(x)$. Then by **Claim 2.10**, $a(x) \star p_\infty(x) = p_\infty(x)$. It follows that $p_\infty(x) = a(x)$.
- $\Leftarrow$ By hypothesis, $p_\infty(x) = a(x)$, and by **Corollary 2.18**, $a(x) \star a(x) = a(x)$. Plugging in the first equation with the second at the appropriate location yields the desired result, and $a(x) \star p_\infty(x) = a(x)$.

□

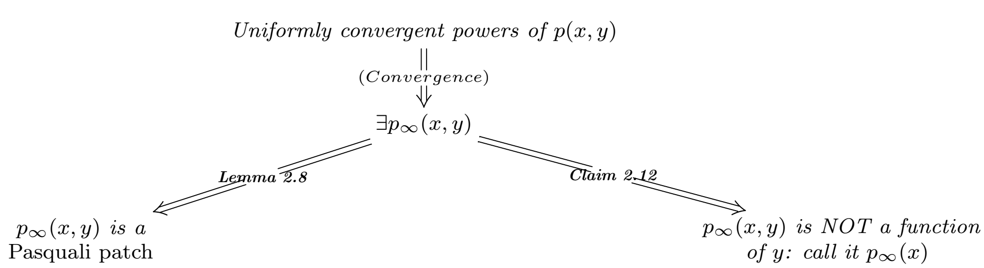
Corollary 2.20. ***(December 3, 2012)*** *$a(x)$ is fixed for a collection of (uniformly convergent) self-powers of the* Pasquali patch *$p(x, y)$ if and only if it is fixed for the limiting* Pasquali patch *(by **Lemma 2.8**) $p_\infty(x)$.*

Proof We have:

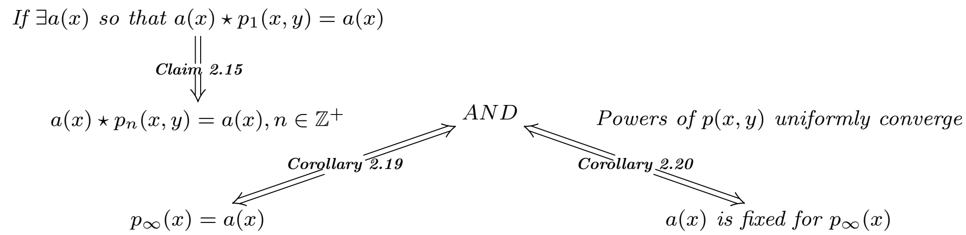
- $\Rightarrow$ By hypothesis, $a(x)$ is fixed for all (uniformly convergent) $p_n(x, y)$. Since they are uniformly convergent, they converge to $p_\infty(x)$. Now, $a(x) \star a(x) = a(x)$ is true by **Corollary 2.18**, and $a(x)$ is the ONLY (probability) probability distribution or *Pasquali patch* that makes the statement true by **Claim 2.17**. Also, $p_\infty(x) \star p_\infty(x) = p_\infty(x)$ by **Lemma 2.8** ($p_\infty(x)$ is a *Pasquali patch*) and **Claim 2.11**. It follows that $p_\infty(x) = a(x)$ with these two pieces of information. Lastly, by **Corollary 2.19** ($\Leftarrow$), we have that $a(x) \star p_\infty(x) = a(x)$.
- $\Leftarrow$ **(Indirect Proof)** Suppose otherwise, that $a(x) \star p_\infty(x) = a(x)$ and $p_\infty(x) = a(x)$ by **Corollary 2.19**, which in turn implies of **Corollary 2.18** that
$$a(x) \star a(x) = a(x) \Rightarrow p_\infty(x) \star p_\infty(x) = p_\infty(x)$$
but $a(x) \star p(x, y) \neq a(x)$ (for some uniformly convergent $p(x, y)$). Then:
$$(a(x) \star p(x, y)) \star a(x) \neq a(x) \star a(x)$$
$$a(x) \star (p(x, y) \star a(x)) \neq a(x) \star a(x)$$
by **Claim 1.8 Associativity of the Star Product**
$$a(x) \star (a(x)) \neq a(x)$$
by **Claim 2.11** and
$$p_\infty(x) \star p_\infty(x) \neq p_\infty(x)$$
contradicts the last implication of the hypothesis.

□

Remark 2.2 (Guiding Diagrams). ***(December 7, 2012)***
The diagrams show at a high-level the most important claims that are relevant to practically calculating the limiting Pasquali patch.
The consequences of uniformly convergent powers are displayed in the following diagram.



Next, fixing $a(x)$ gives us the relationships established below.



Remark 2.3. ***(December 8, 2012)*** *Suppose that, by some sorcery or heuristics or good guess, we have found a candidate probability distribution $a(x)$ so that it is fixed for a given* Pasquali patch *$p(x, y)$. We can hazard the very good guess that the stationary limiting surface is $p_\infty(x) = a(x)$, and this itself is a* Pasquali patch*. It is only a "very good guess" because we have shown this equation to be true for* Pasquali patch *power collections that converge uniformly. In order to be completely certain that $p_\infty(x) = a(x)$, we would have to show that the* Pasquali patch *power collection generated by $p(x, y)$ possesses this property, a task left to, for example, a Weierstrass M-test. The property of uniform convergence becomes unduly burdensome (imagine having to show for each $p(x, y)$ uniform convergence before we can conclude without a shadow of a doubt that the limiting* Pasquali patch *is $p_\infty(x) = a(x)$).*

We are left with two choices: either (1) we prove for ***Construction 2.1*** *or some-such family of* Pasquali patches *the property of uniform convergence, thus limiting ourselves to the study of these particular* Pasquali patches *(seems overly restrictive from my viewpoint), or (2) we weaken the uniform convergence criterion to just convergence (of any sort). In essence, this second argument implies showing that the limit of the integrals of converging* Pasquali patch *powers generated by $p(x, y)$ equals the integral of the limiting* Pasquali patch*: a revision of* ***Lemma 2.8*** *(and subsequent claims that use it). With this, we need only posit convergence and $p_\infty(x) = a(x)$ automatically. This second approach is the one I'm most inclined for and working toward.*

Lemma 2.21. ***(December 3, 2012)*** *If the sequence $p_n(x, y)$ of* Pasquali patch *powers converge ~~uniformly~~ to $p_\infty(x, y)$, then $p_\infty(x, y)$ is a* Pasquali patch.

Proof Suppose we have a collection of converging *Pasquali patch* powers generated by the *Pasquali patch* $p(x, y)$. They must converge to something, so call this $p_\infty(x, y)$. Next look at the sequence

$$\left\{ \int_0^1 p_n(x, y)\, dx \right\}_{n \in \mathbb{Z}^+} = \{1\}_n$$

The sequence is obviously bounded above, but in particular, the least upper bound b is 1. Similarly, the infimum is itself 1. Next let's look at $\int_0^1 p_\infty(x, y)\, dx = c$. Suppose that $c > b$. Then there is an integral of some *Pasquali patch* power that must be greater than 1. But this is a contradiction, since all *Pasquali patch* power integrals are 1. Next suppose $c < b$. The issue is that the sequence $\{1\}_n$ is never less than 1 either. Clearly $c = b$ and we are done. □

Things that change with this amendment:

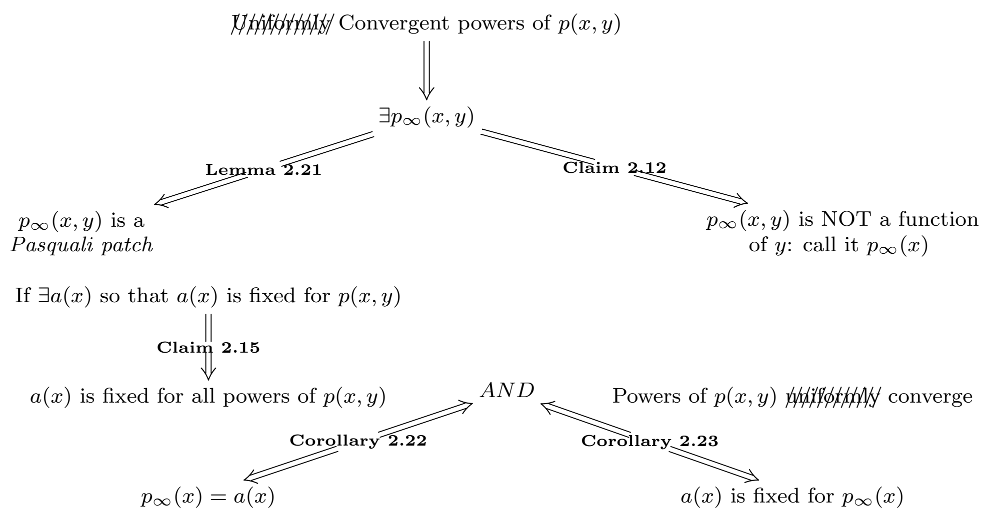
- Proof of **Claim 2.12**. Rather than having Case 2 be the space of ~~uniformly~~ convergent surfaces, it is for convergent surfaces. Also, $p_\infty(x, y)$ is a *Pasquali patch* by **Lemma 2.21**.
- **Corollary 2.19**. Rather than being for ~~uniformly~~ convergent *Pasquali patches*, the corollary holds for convergent *Pasquali patches*: The proof holds with **Lemma 2.21**. This we shall call:

  Corollary 2.22.

- **Corollary 2.20** Again, the corollary holds for convergent *Pasquali patches* and not just ~~uniformly~~ convergent ones. The proof holds with **Lemma 2.21**. This we shall call:

  Corollary 2.23.

- **Remark 2.2** (Guiding Diagrams). Wherever there is a ~~uniform~~ convergence, we can substitute just "convergence." Wherever there is a **Lemma 2.8**, we can substitute it by **Lemma 2.21**. Wherever there is a **Corollary 2.19**, we can change it to **Corollary 2.22**, and wherever there is a **Corollary 2.20**, we can substitute it to **Corollary 2.23**.



Remark 2.4. ***(December 16, 2012)*** *This is why **Example 2.2** actually has a limiting probability distribution. The* Pasquali patch *power collection does not converge uniformly (it converges at the nodes first, e.g.), and yet it stabilizes to the uniform* Pasquali patch *eventually.*

A corollary to all this is:

Corollary 2.24 (Entropy)**.** ***(December 16, 2012)*** *$\exists a(x)$ so that $a(x) \star p(x, y) = a(x)$ if and only if the powers of $p(x, y)$ converge.*

Proof We have:

$\Rightarrow$ Suppose not, that the powers $p(x, y)$ do not converge. One way for this to happen and have $\int_0^1 p(x, y)\, dx = 1$ is if the powers get stuck, for example, in a loop, so that, say,

$$p_{2n+1}(x, y) = p(x, y)$$

and

$$p_{2n}(x, y) = q(x, y)$$

and

$$p(x, y) \neq q(x, y)$$

Then by hypothesis $a(x) \star p_{2n+1}(x, y) = a(x)$ and it must also be true that $a(x) \star p_{2n}(x, y) = a(x)$ for this same $a(x)$ (**Claim 2.15**). This implies that $a(x) \star p(x, y) = a(x)$ and $a(x) \star q(x, y) = a(x)$. Now each even powers and odd powers are independently convergent, $p_{2n+1}(x, y)$ to $p(x, y)$ and $p_{2n}(x, y)$ to $q(x, y)$, which implies that $a(x) = p(x, y)$ in the first case and $a(x) = q(x, y)$ in the second case (**Corollary 2.22**). Other than the fact that both $p(x, y)$ and $q(x, y)$ could be functions of x, y, they are defined to be unequal to each other. We have a contradiction.

Next, let's make the loop larger. Suppose that the powers oscillate in k different surface forms, so that we have $p^1(x, y) \ldots p^k(x, y)$ surfaces after which we return to the original. The above would mean that $p_n(x, y) = p^{n \bmod k}(x, y)$. Each "in-between" power is independently convergent to itself after k more powers, and $a(x) = p^k(x, y)$ due to **Corollary 2.22**. But these surfaces were defined to be different from each other, and so, as before, we face a contradiction.

We show of course the $k + 1$ case. Suppose the powers oscillate in $k + 1$ different surface forms, with

$$p^1(x, y) \ldots p^{k+1}(x, y)$$

surfaces after which we return to the original. Then

$$p_n(x, y) = p^{n \bmod (k+1)}(x, y)$$

with each in-between power independently convergent to itself after $k+1$ more powers, and $a(x) = p^{k+1}(x, y)$ due to **Corollary 2.22** each. Each of these surfaces was different. We have a contradiction.

Now we can make the periodicity as large as we like.

This same argument applies for the case in which the oscillatory pattern is established (independently) to m different surfaces eventually. Suppose that we have $p^1(x,y)\dots p^m(x,y)$ surfaces toward which each periodic sequence $p_{n \bmod m}(x,y)$ converges eventually, with

$$\lim_{n\to\infty} p_{n \bmod m}(x,y) = p^{n \bmod m}(x,y)$$

An argument with respect to $a(x)$ and the surfaces of convergence being unequal leads us to contradict, even inductively, that such a scenario is a possibility. An eventual convergent loop cannot form, no matter how large the loop or how slow or fast each convergence.

We have shown essentially that we cannot have an $a(x)$ and have the *Pasquali patch* powers of $p(x,y)$ oscillate in any shape or form in the long run.

We have not shown the "divergent" situation, in which the value of at least one point $p_n(\overline{x},\overline{y}) \to \infty$. In particular, the product $a(1-\overline{y})\cdot p_n(\overline{x},\overline{y}) \to \infty$, and the value of the integral $\int_0^1 a(1-y)\cdot p_n(\overline{x},y)\,dy \to \infty$ as well. Thus $a(\overline{x})$ diverges and $a(x)$ was never bounded or well-behaved (Contradicting **Conjecture 2.1** regarding $a(x)$). NB: This argument can probably be made more rigorous.

Lastly, since the powers neither stabilize oscillatorily nor can they diverge, they must converge.

$\Leftarrow$ **(December 24, 2012)** Since the powers of $p(x,y)$ converge, they converge to $p_\infty(x)$ by **Claim 2.12**, a function of x alone or constant for all x,y. Then it is certainly true that $p(x,y)\star p_\infty(x) = p_\infty(x)$ by **Claim 2.10**. It must also be true by the same claim that $p_\infty(x)\star p_\infty(x) = p_\infty(x)$. If we substitute one equation into the other, we get $p_\infty(x)\star(p(x,y)\star p_\infty(x)) = p_\infty(x)$. By **Claim 1.8, Associativity of the Star Product**, we can rewrite that as $(p_\infty(x)\star p(x,y))\star p_\infty(x) = p_\infty(x)$. But now the arguments in parenthesis must be equal to $p_\infty(x)$, in other words, $p_\infty(x)\star p(x,y) = p_\infty(x)$. Thus $p_\infty(x)$ is fixed for $p(x,y)$. Let $a(x)$ be exactly this $p_\infty(x)$ and we are done.

□

Remark 2.5. ***(December 24, 2012)*** *It is now evident that, so long as we can find a fixed $a(x)$, we know that the powers of $p(x,y)$ (1) converge, (2) the stationary surface is $p_\infty(x)$, that (3) $a(x) = p_\infty(x)$, and (4) $a(x)$ is fixed for all powers, including the stationary surface itself. Conversely, if we know that $p_\infty(x)$ for some $p(x,y)$, then such is in fact $a(x)$. Our efforts from now on should focus on establishing a mechanism to find such $a(x)$.*

2.6. Fixed Distribution Existence.

Claim 2.25. ***(December 2, 2012)*** *For **Construction 2.1**, $a(x)$ exists provided $B = a(x)\star g_1(y)$ converges and can be solved. Its explicit form is*

$$a(x) = p_\infty(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right)B$$

Proof We are looking for $a(x)$ so that $a(x)\star p(x,y) = a(x)$, with

$$p(x,y) = f_1(x)g_1(y) + f_2(x)\left(\frac{1-g_1(y)F_1}{F_2}\right)$$

Using the definition of the star operator (**Definition 1.1**), this is

$$\int_0^1 a(1-y)\left(f_1(x)g_1(y) + f_2(x)\left(\frac{1-g_1(y)F_1}{F_2}\right)\right)dy = a(x)$$

Expansion results in:

$$a(x) = f_1(x)\int_0^1 a(1-y)g_1(y)\,dy + \frac{f_2(x)}{F_2}\cdot 1 - \frac{F_1}{F_2}f_2(x)\int_0^1 a(1-y)g_1(y)\,dy$$

where we have simplified $\int_0^1 a(1-y)\,dy$ to 1 because the transformation to the y-axis does not change the integral result (it remains a probability distribution).

Rearranging, we have

$$\frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right)\int_0^1 a(1-y)g_1(y)\,dy$$

or

$$a(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right)B = p_\infty(x)$$

(**Remark 2.5**) and derivatives

$$a^i(x) = \frac{f_2^i(x)}{F_2} - \left(\frac{f_2^i(x)F_1}{F_2} - f_1^i(x)\right) B$$

We want to obtain B, but the expression $\int_0^1 a(1-y)g_1(y)\,dy$ has to be clearly defined. We use the tabular method to simplify the integration by parts.

Derivatives	Integrals
$a(1-y)$	$g_1(y)$
$-a'(1-y)$	$G_1^1(y)$
$a(1-y)$	$G_1^2(y)$
$\vdots$	$\vdots$

Viewed from a different vantage-point, we could have

Derivatives	Integrals
$g_1(y)$	$a(1-y)$
$g_1^{'}(y)$	$-A^1(1-y)$
$g_1^{''}(y)$	$A^2(1-y)$
$\vdots$	$\vdots$

Lastly, we have:

$$B = a(1-y)G_1^1(y) + a^{'}(1-y)G_1^2(y) + \ldots\Big|_0^1 = \sum_{i=0}^{\infty} a^i(1-y)G_1^{i+1}(y)\Bigg|_0^1$$

or

$$B = -g_1(y)A^1(1-y) - g_1^{'}(y)A^2(1-y) - \ldots\Big|_0^1 = -\sum_{i=0}^{\infty} g_1^i(y)A^{i+1}(1-y)\Bigg|_0^1$$

If the sum diverges we are stuck, but if the sum converges we are good. □

Corollary 2.26. ***(December 25, 2012)*** Pasquali patches *constructed as by* ***Construction 2.1*** *with (finite) polynomial function choices for $f_1(x)$ and $f_2(x)$ are guaranteed to have a fixed $a(x) = p_\infty(x)$ regardless of (integrable) function choice $g_1(y)$. Similarly,* ***Construction 2.1*** Pasquali patches *with a (finite) polynomial choice for $g_1(y)$ are guaranteed to have such fixed $a(x)$ as well, regardless of (integrable) function choices for $f_1(x)$ and $f_2(x)$.*

Proof Since

$$B = a(1-y)G_1^1(y) + a^{'}(1-y)G_1^2(y) + \ldots\Big|_0^1 = \sum_{i=0}^{\infty} a^i(1-y)G_1^{i+1}(y)\Bigg|_0^1$$

and the derivatives of $a(x)$ are eventually zero (for all subsequent derivatives), the sum itself is finite. Thus, B converges, which implies that **Claim 2.25** applies.

Next, since

$$B = -g_1(y)A^1(1-y) - g_1^{'}(y)A^2(1-y) - \ldots\Big|_0^1 = -\sum_{i=0}^{\infty} g_1^i(y)A^{i+1}(1-y)\Bigg|_0^1$$

and the derivatives of $g_1(y)$ are eventually zero (including all subsequent derivatives), such sum is also finite and B converges. Again, **Claim 2.25** applies. □

Example 2.4. ***(January 16, 2011)*** *In* ***Example 2.1****, we had the* Pasquali patch

$$p(x,y) = x^2y^3 + x\left(2 - \frac{2y^3}{3}\right)$$

with $f_1(x) = x^2$, $f_2(x) = 2x$, $g_1(y) = y^3$. This implies by ***Claim 2.25*** *that*

$$a(x) = p_\infty(x) = 2x - \left(\frac{2x}{3} - x^2\right) B$$

with derivatives

$$a^{'}(x) = 2 - \left(\frac{2}{3} - 2x\right) B$$

$$a^{''}(x) = 2B$$

Specifically,

$$\begin{array}{ccc} a(1) = 2 + \frac{B}{3} & and & a(0) = 0 \\ a^{'}(1) = 2 + \frac{4B}{3} & and & a^{'}(0) = 2 - \frac{2B}{3} \\ a^{''}(1) = 2B & and & a^{''}(0) = 2B \end{array}$$

Next, we want to calculate B:

$$B = \cancel{a(0)G_1^1(1)} + a^{'}(0)G_1^2(1) + a^{''}(0)G_1^3(1) - \cancel{\left(a(1)G_1^1(0) + a^{'}(1)G_1^2(0) + a^{''}(1)G_1^3(0)\right)}$$

The parenthetical part dies because integrals of y^3 evaluated at 0 vanish. So does the first term since $a(0) = 0$. So we are left with:

$$B = a^{'}(0)G_1^2(1) + a^{''}(0)G_1^3(1) = \frac{6-2B}{3} \cdot \frac{1}{20} + 2B \cdot \frac{1}{120}$$

which solves

$$B = \frac{6}{61}$$

Thus, we have that

$$a(x) = p_\infty(x) = 2x - \left(\frac{2x}{3} - x^2\right)\frac{6}{61} = \frac{6x^2}{61} + \frac{118x}{61}$$

*For a consistency check, by **Lemma 2.21** this is a* Pasquali patch *and therefore*

$$\int_0^1 \left(\frac{6x^2}{61} + \frac{118x}{61}\right) dx = 1$$

can be verified to indeed be the case.

Example 2.5. ***(January 16, 2011)***

*As in **Example 2.2**, take the* Pasquali patch $p(x, y) = 1 - \cos(2\pi x)\cos(2\pi y)$. *To establish the steady state surface, we write*

$$\int_0^1 a(1-y)\left(1 - \cos(2\pi x)\cos(2\pi y)\right) dy = a(x)$$

or, explicitly,

$$\int_0^1 a(1-y)\, dy - \cos(2\pi x) \underbrace{\int_0^1 a(1-y)\cos(2\pi y)\, dy}_{B} = a(x)$$

The first integral adds up to 1 by hypothesis, where the second one is zero after integrating by parts:

$$\begin{array}{cc} Derivatives & Integrals \\ a(1-y) & \cos(2\pi y) \\ -a'(1-y) & \frac{\sin(2\pi y)}{2\pi} \\ a''(1-y) & \frac{-\cos(2\pi y)}{4\pi} \\ -a'''(1-y) & \frac{-\sin(2\pi y)}{8\pi} \\ \vdots & \vdots \end{array}$$

so we have

$$\left(\frac{a(1-y)\sin(2\pi y)}{2\pi} - \frac{a'(1-y)\cos(2\pi y)}{4\pi} - \frac{a''(1-y)\sin(2\pi y)}{8\pi} + \dots\right)\Bigg|_0^1$$

or

$$-\frac{a'(0)}{4\pi} + \frac{a'''(0)}{16\pi} - \frac{a^v(0)}{64\pi} + \dots - \left(-\frac{a'(1)}{4\pi} + \frac{a'''(1)}{16\pi} - \frac{a^v(1)}{64\pi} + \dots\right)$$

Next, we are left with

$$B = \frac{a'(1)}{4\pi} - \frac{a'(0)}{4\pi} + \frac{a'''(0)}{16\pi} - \frac{a'''(1)}{16\pi} + \frac{a^v(1)}{64\pi} - \frac{a^v(0)}{64\pi} + \dots$$

and also with

$$\begin{array}{rcl} a(x) & = & 1-\cos(2\pi x)B \\ a'(x) & = & 2\pi\sin(2\pi x)B \\ a''(x) & = & 4\pi\cos(2\pi x)B \\ a'''(x) & = & -8\pi\sin(2\pi x)B \\ \vdots & \vdots & \vdots \end{array}$$

To show this thoroughly, we should prove by induction that every odd derivative of $a(x)$ *contains a* sin *term (or we can attempt an argument by periodicity of the derivative, as we do), and so evaluating such at* 0 *and at* 1 *literally causes the term to vanish, and leaving us with the fact that* $B = 0$ *and that* $a(x) = 1$. *Therefore* $p_\infty(x, y) = a(x) = 1$.

Remark 2.6. *Practically speaking, all functions* $f(x), g(y)$ *with Taylor polynomial representations in the domain* $[0, 1]$ *will converge (just truncate them at the appropriate precision, e.g.).*

2.7. **Probability Distribution Transformations via *Pasquali Patch* Power Collections.**

Claim 2.27. ***(January 12, 2013)*** *Take a* Pasquali patch *power collection union its limiting surface,* $\mathbb{P} \cup p_\infty(x) = \mathbb{P}^\infty$*, and a well-behaved, bounded probability distribution* $c(x)$*, with* $c\colon [0, 1] \to \mathbb{R}^+ \cup \{0\}$*, and* $\int_0^1 c(x)\,dx = 1$. *Then* $c(x) \star \mathbb{P}^\infty = \mathbb{C}$ *is a collection of transformed probability distributions of* x *(explicit or not).*

Proof by Induction The proof consists of two cases.

Case 1. The generator of $\mathbb{P}^\infty$ is a *Pasquali patch* function of x alone (explicitly or constant). Then, by **Claim 2.11** all *Pasquali patches* in the collection are functions of x (explicitly or constant), and in fact all powers in $\mathbb{P}$ equal $p(x)$, including $p_\infty(x)$ (**Case 1 of Claim 2.12**). Begin with

$$c(x) \star P^1 = c(x) \star p(x) = p(x)$$

by **Claim 2.10**. Assume that $c(x) \star P^k = p(x)$. Then

$$c(x) \star P^{k+1} = c(x) \star (P^1 \star P^k) = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star p(x) = p(x) \star p(x) = p(x)$$

We used **Claim 1.9 Powering Symmetry** to exchange the order of the powers, **Claim 1.8 Associativity of the Star Product**, and again by **Claim 2.10** the last equality holds. Furthermore, since $p_\infty(x) = p(x)$, $c(x) \star p_\infty(x) = c(x) \star p(x) = p(x)$. This last part is justified by **Claim 2.10** once more. Thus, $c(x) \star \mathbb{P}^\infty = \mathbb{C} = \{p(x)\}$ and all functions are explicit functions of x or constant as we wanted to show. By **Claim 2.13**, the collection is made up of probability distributions on $[0, 1]$.

Case 2. The generator of $\mathbb{P}^\infty$ is a *Pasquali patch* function of y explicitly. Begin by

$$c(x) \star P^1 = \int_0^1 c(1-y)p(x, y)\,dy = c_1(x)$$

Since we are integrating with respect to y we can see that this function $c_1(x)$ is either explicitly function of x or constant. Next assume that $c(x) \star P^k = c_k(x)$ is an explicit function of x or constant. Then

$$c(x) \star P^{k+1} = c(x) \star P^{1+k} = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star P^1 = c_k(x) \star P^1$$

This last part is $\int_0^1 c_k(1-y)p(x, y)\,dy$, which, by virtue of integrating in terms of y yields an explicit function of x or constant. Now take $c(x) \star p_\infty(x) = p_\infty(x)$ by **Claims 2.12** and **2.10**. Thus, all functions in the collection $c(x) \star \mathbb{P}^\infty = \mathbb{C}$ are explicit functions of x or constant. Finally, by **Claim 2.13**, the collection is made up of probability distributions on $[0, 1]$.

□

Claim 2.28. ***(January 12, 2013)*** *Let* $c(x) \star P^m = c_m(x)$*, the* m*th transform of* $c(x)$ *via the* m*th power of* $p(x, y)$. *An equivalent way to obtain the* m*th transform is by* $c_{m-1}(x) \star P^1 = c_m(x)$.

Proof by Induction Begin by $c(x) \star P^1 = c_1(x)$ using the first statement and $c_0(x) \star P^1 = c_1(x)$ using the second statement. Clearly the two statements are equivalent. Next, assume that the statement holds for the kth transform, so that $c(x) \star P^k = c_k(x)$ and $c_{k-1}(x) \star P^1 = c_k(x)$. Now $c(x) \star P^{k+1} = c_{k+1}(x)$ by definition. An equivalent way to write this is

$$c(x) \star P^{1+k} = c(x) \star (P^k \star P^1) = (c(x) \star P^k) \star P^1 = c_k(x) \star P^1$$

Here we have used **Claim 1.8 Associativity of the Star Product**. Thus $c_k(x) \star P^1 = c_{k+1}(x)$, and we are done. □

Claim 2.29. ***(January 13, 2013)*** *The collection* $\mathbb{C}$ *converges to* $p_\infty(x)$.

Indirect Proof 1 Suppose $\lim_{m\to\infty} c_m(x) \neq p_\infty(x)$. Then

$$\left[\lim_{m\to\infty} c_m(x)\right] \star p_\infty(x) \neq p_\infty(x) \star p_\infty(x)$$

and

$$p_\infty(x) \neq p_\infty(x) \star p_\infty(x)$$

is a contradiction of **Claim 2.11** or **Claim 2.10**. Thus $\lim_{m\to\infty} c_m(x) = p_\infty(x)$. □

Indirect Proof 2 Suppose that the sequence of probability distributions $c_m(x)$ diverge, that

$$\lim_{m\to\infty} c_m(\overline{x}) = \infty$$

for some $\overline{x}$. This would have to mean that $\lim_{m\to\infty} \int_0^1 c(1-y)p_m(\overline{x},y)\,dy = \infty$. But then the sequence $p_m(\overline{x},y)$ would have to be divergent, because $c(x)$ is chosen to be well-behaved and bounded. We have a contradiction, because the sequence $p_m(x,y)$ is well-behaved and bounded by virtue of being *Pasquali patch* powers with bounded first-power generator and fixed volume. Thus $\lim_{m\to\infty} c_m(x)$ converges.

Next suppose the sequence of probability distributions $c_m(x)$ does not converge to $p_\infty(x)$ but to some other probability distribution in the collection $\mathbb{C}$, say $c_k(x)$. The issue is this isn't an accumulation probability distribution, since we can generate $c_{k+1}(x)$ by starring by P^1 on the right (**Claim 2.28**). Lastly, say we choose a probability distribution outside of the collection $\mathbb{C}$, in some other collection $\mathbb{D}$. If we choose $d_k(x)$, this isn't an accumulation probability distribution for $\mathbb{C}$ because it was generated by *Pasquali patch* $q(x,y)$, and also $d_{k+1}(x)$ can be generated by right-starring by Q^1. Then pick $p^d_\infty(x)$, the limiting probability distribution in $\mathbb{D}$. There's just no way to tie such to the generator collection $\mathbb{P}^\infty$, since it is an accumulation probability distribution for $\mathbb{D}$, not for $\mathbb{C}$. The only choice left (that makes sense) is $p^c_\infty(x)$. □

Corollary 2.30. ***(January 13, 2013)***

$$\lim_{m\to\infty} c_m(x) = p_\infty(x)$$

if and only if

$$\lim_{m\to\infty} [c(x) \star P^m] = c(x) \star \left[\lim_{m\to\infty} P^m\right]$$

In other words, we can pull the limiting process under the integral.

Proof We have:

⇒ By definition, $c(x) \star P^m = c_m(x)$. Then

$$\lim_{m\to\infty} c_m(x) = \lim_{m\to\infty} [c(x) \star P^m] = p_\infty(x)$$

by hypothesis.

Next,

$$\lim_{m\to\infty} [c(x) \star P^m] = \lim_{m\to\infty} \int_0^1 c(1-y)p_m(x,y)\,dy = p_\infty(x)$$

An equivalent expression can be found through **Claim 2.10**:

$$p_\infty(x) = \int_0^1 c(1-y)p_\infty(x)\,dy$$

This is,

$$p_\infty(x) = \int_0^1 c(1-y)\left[\lim_{m\to\infty} p_m(x,y)\right]\,dy$$

In other words, we now have

$$p_\infty(x) = c(x) \star \left[\lim_{m\to\infty} P^m\right]$$

Putting these two expressions together yields the desired result.

⇐ Take

$$c(x) \star \left[\lim_{m\to\infty} P^m\right] = c(x) \star p_\infty(x) = p_\infty(x)$$

by definition of the limiting surface and **Claim 2.10**. The alternative way to write this by the stated equality in the hypothesis is:

$$\lim_{m\to\infty} [c(x) \star P^m] = p_\infty(x)$$

In turn, this can be written as $\lim_{m\to\infty} c_m(x) = p_\infty(x)$ by definition and we are done.

□

Corollary 2.31. ***(January 15, 2013)***

$$\lim_{m\to\infty}\int_0^1 p_m(x,y)\,dy = p_\infty(x) = \lim_{m\to\infty} p_m(x,y)$$

Proof Take $c(x) = u(x)$, the uniform probability distribution, which equals 1 for all x in the domain. We can then state by **Corollary 2.30** that

$$\lim_{m\to\infty}[1 \star P^m] = 1 \star \left[\lim_{m\to\infty} P^m\right]$$

which is equivalent to:

$$\lim_{m\to\infty}\int_0^1 p_m(x,y)\,dy = \int_0^1 \lim_{m\to\infty} p_m(x,y)\,dy$$

which, in turn, yields

$$\lim_{m\to\infty}\int_0^1 p_m(x,y)\,dy = p_\infty(x) = \lim_{m\to\infty} p_m(x,y)$$

□

Remark 2.7. ***(January 15, 2013)*** *We now have that*

$$\lim_{m\to\infty}\int_0^1 p_m(x,y)\,dx = 1$$

*by **Definition 2.1** and **Claim 2.4 Closure of*** **Pasquali Patches** *and*

$$\lim_{m\to\infty}\int_0^1 p_m(x,y)\,dy = p_\infty(x)$$

*by **Corollary 2.31**.*

Corollary 2.32. ***(January 13, 2013)***

$$\lim_{m\to\infty} c_m(x) = p_\infty(x)$$

if and only if

$$\lim_{m\to\infty}\left[c_{m-1}(x) \star P^1\right] = \left[\lim_{m\to\infty} c_{m-1}(x)\right] \star P^1$$

Proof This follows from **Claim 2.28**. □

Corollary 2.33. ***(January 13, 2013)***

$$\lim_{m\to\infty} c_m(x) = p_\infty(x)$$

if and only if

$$c(x) \star \left[\lim_{m\to\infty} P^m\right] = \left[\lim_{m\to\infty} c_{m-1}(x)\right] \star P^1$$

In other words,

$$c(x) \star p_\infty(x) = c_\infty(x) \star P^1$$

Proof This follows from **Corollary 2.30** and **Corollary 2.32**. □

2.8. More on *Pasquali Patch* Powers and Limiting Surfaces.

Remark 2.8. ***(January 13, 2014)*** *The following claims and proofs are applicable to* Pasquali patches *as they are to Markov matrices, where we shift from a function point-of-view to a matricial point-of-view using in one the star product and in the other normal matrix multiplication.*

Claim 2.34. ***(January 13, 2014)*** *Take the countably infinite collections of Pasquali patch self-powers $\mathbb{P} = \{P^1, P^2, \ldots, P^j, \ldots\}_{j\in\mathbb{Z}^+}$ and $\mathbb{Q} = \{Q^1, Q^2, \ldots, Q^k, \ldots\}_{k\in\mathbb{Z}^+}$ with stationary limiting P^∞ and Q^∞ respectively. If $\mathbb{P}$ and $\mathbb{Q}$ coincide at (m,n) so that $P^m = Q^n$ ($\frac{n}{m} \in \mathbb{Z}^+$) then $P^\infty = Q^\infty$.*

Proof The infinite collections coincide at (m,n) for $m, n, \frac{n}{m} \in \mathbb{Z}^+$, thus we have that $P^m = Q^n$. By **Claim 1.13** we have that $\mathbb{P}_m \subset \mathbb{Q}_n$. Now the limit $\lim_{n\to\infty} Q_n = Q^\infty$ by hypothesis of the existence of the stationary patch, and since $\mathbb{P}_m$ is a subset of $\mathbb{Q}_n$, it follows that $\lim_{m\to\infty} P_m = Q^\infty$. We also know that $\lim_{m\to\infty} P_m = P^\infty$ however (again by hypothesis of the existence of the stationary patch), so it must therefore be true that $P^\infty = Q^\infty$. □

The reverse claim is not (always) true:

Claim 2.35 (False Claim)**.** ***(January 13, 2014)*** *If $P^\infty = Q^\infty$, then $\mathbb{P}$ and $\mathbb{Q}$ coincide at (m,n) so that $P^m = Q^n$ ($\frac{n}{m} \in \mathbb{Z}^+$).*

Disproof Take the collection of *Pasquali patches* $\mathbb{P} = \{P^1 = 1, P^2 = 1, \ldots, P^j = 1, \ldots\}_{j\in\mathbb{Z}^+}$ with $P^\infty = 1$, and any collection of *Pasquali patches* $\mathbb{Q} = \{Q^1, Q^2, \ldots, Q^k, \ldots\}_{k\in\mathbb{Z}^+}$ so that the generator $Q^1 = q(x, y)$ is an explicit function of y and $Q^\infty = 1$, as in **Example 2.2**. By **Claim 2.9**, all elements in the collection $\mathbb{Q}$ are explicit functions of y (even as Q^∞ is not by **Claim 2.12**). Clearly, no $P \in \mathbb{P}$ is equal to an element $Q \in \mathbb{Q}$ (all P are constant where all Q vary with y), yet they have the same stationary patch. □

2.8.1. *Propagation of zeros.*

Claim 2.36. ***(February 12, 2015)*** *Take a* Pasquali patch $p(x, y)$*, so that for all the elements* a_j *in any of the open, disjoint sets* $A_j \subset [0, 1]$ *the* Pasquali patch *evaluates to zero for any choice of* $y \in [0, 1]$*. That is,*

$$p(a_j, y) = 0, \forall a_j \in A_j$$

Then for all elements in the collection $\mathbb{C} = c(x) \star \mathbb{P}^\infty$*, except possibly the first, we have that*

$$c_n(a_j) = 0$$

In particular,

$$c_\infty(a_j) = p_\infty(a_j) = 0$$

Proof by Induction Pick an open set A_j with elements a_j. From the hypothesis, we have that

$$p(a_j, y) = 0, \forall a_j \in A_j$$

Take the probability distribution $c_1(x)$. In order to generate $c_2(x)$, we must star it by $p(x, y)$. Thus

$$c_2(x) = c_1(x) \star p(x, y) = \int_0^1 c_1(1 - y) \cdot p(x, y)\, dy$$

Notice $x \in [0, 1]$ ranges through $A_j \subset [0, 1]$, and will take on the values a_j. Thus, when x is in the set A_j the integral is

$$c_2(a_j) = \int_0^1 c_1(1 - y) \cdot p(a_j, y)\, dy = \int_0^1 c_1(1 - y) \cdot 0\, dy = 0$$

Assume this works for the kth transform, so that

$$c_k(a_j) = 0$$

Then

$$c_{k+1}(a_j) = \int_0^1 c_k(1 - y) \cdot p(a_j, y)\, dy = \int_0^1 c_k(1 - y) \cdot 0\, dy = 0$$

Thus we have shown that

$$c_n(a_j) = 0$$

Observe

$$\lim_{n\to\infty} c_n(a_j) = \lim_{n\to\infty} 0 = 0$$

But

$$\lim_{n\to\infty} c_n(a_j) = c_\infty(a_j) = p_\infty(a_j) = 0$$

as we wanted to show. □

Claim 2.37. ***(February 12, 2015)*** *Take a* Pasquali patch $p(x, y)$*, so that for all the elements* a_j *in any of the open, disjoint sets* $A_j \subset [0, 1]$ *the* Pasquali patch *evaluates to zero for any choice of* $y \in [0, 1]$*. That is,*

$$p(a_j, y) = 0, \forall a_j \in A_j$$

Then

$$p_n(a_j, y) = 0$$

and in particular $p_\infty(a_j) = 0$*. Succinctly,* $\mathbb{P}^\infty(a_j) = 0$*.*

Proof 1 by Induction This follows the same recipe as **Claim 2.36**, but using the definition of the star product on powers of $p(x, y)$. Notice an important distinction that *all Pasquali patches* in the collection inherit the zeros (in the previous claim the first probability distribution $c_1(x)$ could be excepted). □

Proof 2 The $c_n(x)$ can also be written as

$$c_n(x) = c_1(x) \star p_n(x, y)$$

by **Claim 2.28**. Take the results of **Claim 2.36**, that is, $c_n(a_j) = 0$ except possibly $c_1(a_j)$ (but including $c_\infty(a_j)$). The only way to make sure that such $c_n(a_j)$ are in every circumstance zero is by requiring $p_n(a_j, y) = 0$ for all positive integer n. Since $p_\infty(x) = c_\infty(x)$, and $c_\infty(a_j)$ was shown to be zero, it follows that $p_\infty(a_j) = 0$ and thus the whole set $\mathbb{P}^\infty(a_j) = 0$. □

2.9. **Even More on *Pasquali Patches* and Limiting Surfaces.**

2.9.1. *Preparatory Claims.*

Claim 2.38. ***(May 5, 2014)*** *Take $f_i(x) = (i+1)x^i$ with $i \in \mathbb{Z}^+ \cup \{0\}$. Then $\int_0^1 f_i(x)\,dx = 1, \forall i$.*

Proof by Induction Using the definition of integration of powers of x, we show that $\int_0^1 f_0(x)\,dx = 1$. The expression equals

$$\int_0^1 x^0\,dx = \int_0^1 1\,dx = x\,|_0^1 = 1$$

We assume that the kth element $\int_0^1 f_k(x)\,dx = 1$ although we readily know by the definition of integration that such is true, since

$$\int_0^1 (k+1)x^k\,dx = x^{k+1}\,|_0^1 = 1^{k+1} = 1$$

The exact same definition argument applies to the $k+1$th element and

$$\int_0^1 (k+2)x^{k+1}\,dx = x^{k+2}\,|_0^1 = 1^{k+2} = 1$$

□

Claim 2.39. ***(May 5, 2014)*** *The functions $f_i(x) = (i+1)x^i$ with $i \in \mathbb{Z}^+ \cup \{0\}$ are* Pasquali patches.

Proof A *Pasquali patch* is a function $p(x, y)$ so that $\int_0^1 p(x, y)dx = 1$ by **Definition 2.1**. Let $p(x, y) = f_i(x)$. Since by **Claim 2.38** $\int_0^1 f_i(x)\,dx = 1, \forall i = 1 \ldots n$, then applying the definition means $f_i(x) = (i+1)x^i$ are Pasquali patches $\forall i \in \mathbb{Z}^+ \cup \{0\}$. □

Claim 2.40. ***(May 5, 2014)*** *The finite polynomial $g(x) = \sum_{i=0}^n (i+1)x^i$ converges in area from $[0, 1]$ to $n+1$.*

Proof We are looking for

$$\int_0^1 \sum_{i=0}^n (i+1)x^i\,dx$$

The sum is finite so it converges, and there is no issue exchanging the order of the sum and integral. Thus:

$$\sum_{i=0}^n \int_0^1 (1+i)x^i\,dx = \sum_{i=0}^n \left(x^{i+1}\,|_0^1\right) = \sum_{i=0}^n 1^{i+1} = \sum_{i=0}^n 1 = n+1$$

□

Claim 2.41. ***(May 5, 2014)*** *Pick n functions from the pool of $f_i(x) = (i+1)x^i$. For example, pick $f_3(x), f_5(x)$, and $f_7(x)$. Create the function $h(x) = \sum_i f_i(x)$. Then $\int_0^1 h(x)\,dx = n$.*

Proof by Induction Since by **Claim 2.39** all $f_i(x)$ are *Pasquali patches*, it follows their integral is 1 in the interval (**Claim 2.38**). Picking 1 function from the pool thus gives an integral of 1 in the interval. Suppose that picking k functions gives k units at the integral in the interval. Now pick $k+1$ functions. The first k functions give k units at the integral in the interval, and the 1 additional function contributes 1 unit at the integral in the interval. Thus $k+1$ functions contribute $k+1$ units at the integral in the interval. □

Corollary 2.42. ***(May 5, 2014)*** *The infinite polynomial $a(x) = \sum_{i=0}^\infty (i+1)x^i$ diverges in area in the interval from $[0, 1]$.*

Proof Take

$$\int_0^1 \left(\lim_{n\to\infty} \sum_{i=0}^{n} (1+i)x^i \right) dx = \lim_{n\to\infty} \int_0^1 \sum_{i=0}^{n} (1+i)x^i \, dx$$

Here exchanging the order of limit and integral is justified by the fact that, term-wise, the integral converges. Next

$$\lim_{n\to\infty} n+1 = \infty$$

Here the second to last step is justified by **Claim 2.40**.

Corollary 2.43. ***(May 5, 2014)*** *The infinite polynomial $a(x) - h(x)$ diverges in area in the interval from $[0, 1]$.*

Proof Take the limit

$$\lim_{n\to\infty} [a(x) - h(x)]$$

Taking n to infinity applies to $a(x)$ only which we know diverges by **Corollary 2.42**. The same limit has no effect on $h(x)$ as the sum it is composed of is finite and adds up to an integer constant, say m. We conclude that any infinite collection of terms of $f_i(x)$ diverges, even when a finite number of them may be absent from the sum. □

Corollary 2.44. ***(May 5, 2014)*** *The infinite polynomial $a(x) - b(x)$ diverges in area in the interval from $[0, 1]$ with $a(x), b(x)$ are infinite polynomials constructed by sums of functions picked from the pool $f_i(x) = (i + 1)x^i$ and with no repetitions. (Note that the difference of these two infinite polynomials must also be infinite).*

Proof Since the $a(x) - b(x)$ is an infinite polynomial, the integral of such will be an infinite string of ones since the functions it contains are $f_i(x)$ and these are *Pasquali patches* (**Claim 2.39**) and there are no repetitions. Such infinite sum of ones clearly diverges. □

Remark 2.9. ***(May 5, 2014)*** *We can view what we have learned in the claims from a slightly different vantage point. Create the infinite identity matrix*

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Next create the following polynomial differential vector

$$D = \begin{bmatrix} 1 \\ 2x \\ 3x^2 \\ \vdots \end{bmatrix}$$

It is clear that

$$\int_0^1 I_i \cdot D \, dx = 1$$

for all rows i of I. We can omit the little i because this definition applies to all rows and:

$$\int_0^1 I \cdot D \, dx = \int_0^1 D \, dx = \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix} = \mathbf{1}$$

*This of course summarizes **Claim 2.38** and **Claim 2.39**. Next, define the matrix J consisting of rows which are finite sums of rows of I (so that each row of J consists of a finite number of ones at any position, namely n such coming from n picked rows of I). **Claim 2.40** and **Claim 2.41** are summarized in the statement*

$$\int_0^1 J \cdot D \, dx = S$$

where S is the vector consisting of the sum of the rows of J, which, since it is made up of a finite number of ones at each row, adds up to a constant integer at each row:

$$S = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \end{bmatrix}$$

Finally, the corollaries can be summarized in the statement in which we create a matrix K consisting of rows with a finite number of zeroes (and an infinite number of ones) or an infinite number of zeroes but an infinite number of ones as well. It is clear then that

$$\int_0^1 K \cdot D \, dx = \infty$$

Remark 2.10. ***(May 5, 2014)*** *The cool thing about this notation is that it gives us power to conclude several interesting things. For example, scaling of matrices I and J as by a constant t shows convergence at the integral in the interval $[0, 1]$ of every one of the scaled sums represented by the rows of such matrices.*

Thus:

Corollary 2.45. ***(May 5, 2014)*** *Let $I^* = t \cdot I$ and $J^* = t \cdot J$ with t is a scaling factor. Then the area of each of the infinitely many polynomials represented by the matrices I^*, J^* dot D in the interval from 0 to 1 converge.*

Proof On the one hand, we have

$$\int_0^1 I^* \cdot D \, dx = \int_0^1 t \cdot I \cdot D \, dx = t \left(\int_0^1 I \cdot D \, dx \right) = \mathbf{t}$$

On the other hand,

$$\int_0^1 J^* \cdot D \, dx = \int_0^1 t \cdot J \cdot D \, dx = t \left(\int_0^1 J \cdot D \, dx \right) = t \cdot S = \begin{bmatrix} t \cdot n_1 \\ t \cdot n_2 \\ \vdots \end{bmatrix}$$

□

Remark 2.11. ***(May 5, 2014)*** *Next consider the uncountably (see **Lemma 2.47**) infinite-matrix formed by convergent sequences (at the sum) at each row,*

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1}{2^2} & \frac{1}{3^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Depicted is the reciprocals of squares which we know converges at the sum (Basel problem), simply for illustration, but all convergent sequences would be in an ith row of A. We have

$$\int_0^1 A_i \cdot D \, dx = \sum_j a_{i,j}$$

is convergent by definition. The cool thing is we can easily prove in one swoop that all sequences that are scaled will also converge at the sum (and the infinite polynomials with coefficients $A \cdot D$ have converging area in the interval from 0 to 1).

Corollary 2.46. ***(May 5, 2014)*** *Let $A^* = t \cdot A$ with t is a scaling factor $t \in \mathbb{R}$. Then the area of each of the infinitely many polynomials represented by the matrix entries of $A^* \cdot D$ in the interval from 0 to 1 converge.*

Proof We have

$$\int_0^1 A_i^* \cdot D \, dx = \int_0^1 t \cdot A_i \cdot D \, dx = t \left(\int_0^1 A_i \cdot D \, dx \right) = t \cdot \sum_j a_{i,j}$$

for all i.

Lemma 2.47. ***(September 8, 2014)*** *The matrix A as we defined it has uncountably infinite rows.*

Proof This follows from the fact that all sequences that are scaled will also converge at the sum, that is, by **Corollary 2.46**. Since any fixed, real t is represented in the expression $t \cdot \sum_j a_{i,j}$, with the expression being one that converges, and t is in $\mathbb{R}$, with the set $\mathbb{R}$ is uncountably infinite, then uncountably infinite sequences must be represented in the matrix A. Thus A contains uncountably infinite rows. □

All of these small and obvious observations lead to the following Claim.

2.9.2. *Classifying* Pasquali Patches *that are Functions of x Alone.*

Claim 2.48 (The Grand Classification Theorem: a General and Absolutely Complete Classification of *Pasquali Patches* which are Functions of x Alone)**.** ***(May 5, 2014)*** *All* Pasquali patches *which are functions of x alone (and therefore possible limiting surfaces) take the form*

$$p(x) = \frac{A_i \cdot D}{\sum_j a_{i,j}}$$

with $\sum_j a_{i,j} \neq 0$.

Proof We have that, since such $p(x)$ is a *Pasquali patch*, it must conform to **Definition 2.1**. Thus

$$\int_0^1 p(x)\,dx = \int_0^1 \frac{A_i \cdot D}{\sum_j a_{i,j}}\,dx = \frac{\int_0^1 A_i \cdot D\,dx}{\sum_j a_{i,j}} = \frac{\sum_j a_{i,j}}{\sum_j a_{i,j}} = 1$$

shows this is indeed the case. To show that *all Pasquali patches* that are functions of x alone are of the form of $p(x)$, we argue by contradiction. Suppose that there is a *Pasquali patch* that is a function of x alone which does not take the form of $p(x)$. It couldn't possibly be one such that is a finite polynomial, since A_i was defined to be that matrix formed by all convergent sequences at the sum at each row and it can be scaled any which way we like, and this includes sequences with a finite number of nonzero coefficients. But now it couldn't be any infinite polynomial either, by the same definition of A_i which includes infinite sequences so that $\sum_j a_{i,j}$ is convergent. Thus it must be a polynomial formed by dotting divergent sequences (at the sum), but all such have been happily excluded from the definition of A. □

// issue with remark, pasquali patch includes def that all values must be positive real

Remark 2.12. ***(May 5, 2014)*** *Thus, EVERY convergent series has an associated* Pasquali patch *which is solely a function of x, and vice versa, covering the totality of the* Pasquali patch *functions of x universe and the convergent series universe bijectively once we take into account equivalence classes (due to the normalization factor of the polynomial part). These equivalence classes will be described presently.*

Remark 2.13. ***(May 5, 2014)*** *Notice how the definition takes into account Taylor polynomial coefficients (thus all analytic functions are included) and those that are not (even those that are as yet unclassified), and all sequences which may be scaled by a factor as well.*

Claim 2.49. ***(May 5, 2014)*** *Let $f(x)$ is Maclaurin-expandable so that*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

Then

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{(n+1)!} = \int_0^1 f(x)\,dx$$

Proof

$$\int_0^1 f(x)\,dx = \int_0^1 A_i \cdot D\,dx$$

for some ith row of A. Such a row would have to be of form

$$A_i = \begin{bmatrix} f(0) & \dots & \frac{f^n(0)}{n!(n+1)} & \dots \end{bmatrix}$$

Then the integral

$$\int_0^1 A_i \cdot D\,dx = \sum_j a_{i,j} = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!(n+1)} = \sum_{n=0}^{\infty} \frac{f^n(0)}{(n+1)!}$$

□

Remark 2.14. ***(May 5, 2014)*** *Notice that all Maclaurin-expandable functions converge in area (have stable area) in the interval from 0 to 1, a remarkable fact.*

Example 2.6. ***(May 5, 2014)*** *Take*

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By applying ***Claim 2.49****, it follows that*

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \int_0^1 e^x \, dx = e - 1$$

Remark 2.15. ***(May 5, 2014)*** *Now we have a happy way to construct (any and all)* Pasquali patches *which are functions of x alone, merely by taking a sequence which is convergent at the sum. From a different vantage point, we have a way to describe* any and all *probability distributions in the interval* $[0, 1]$ *merely by specifying a sequence convergent at the sum!*

Remark 2.16. ***(May 5, 2014)*** *Quantum mechanically, we now know all possible shapes that a stationary (limiting) eigen wavevector can take.*

Remark 2.17. ***(May 5, 2014)*** *This gives us extraordinary power to calculate convergent sums via integration. It also gives us extraordinary power to express any number as an infinite sum, for example.*

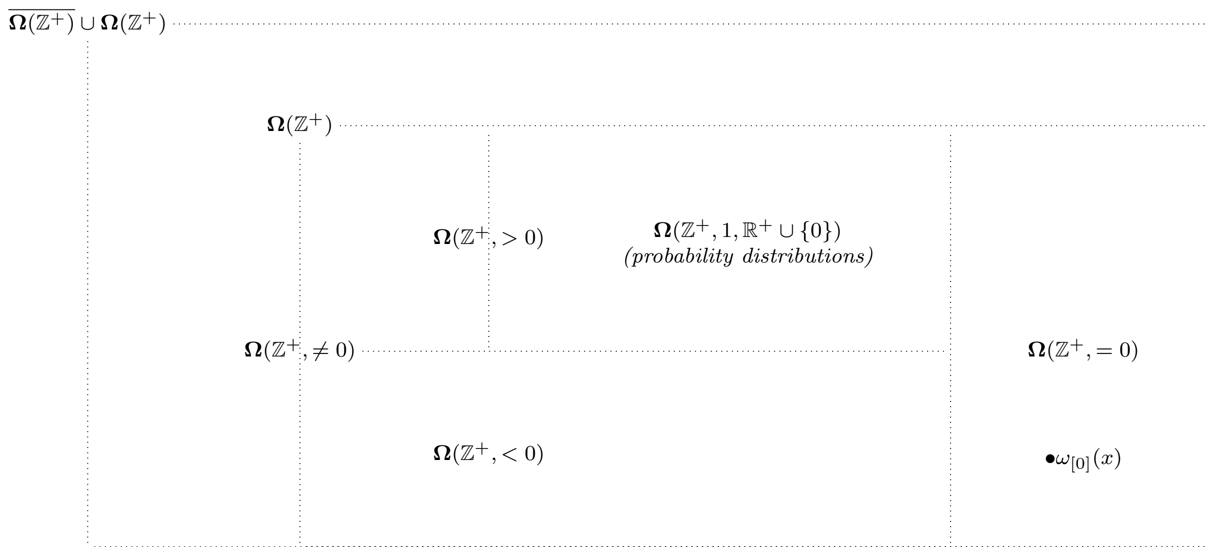
3. Polynomial Functions of x with convergent area in the interval $[0, 1]$

Definition 3.1. ***(August 23, 2014)***

- *Let $\mathbf{\Omega}(\mathbb{Z}^+)$ is the set of all polynomials with convergent area in the interval* $[0, 1]$*, with discrete matricial entries for A, indexed by $\mathbb{Z}^+$.*
- *Let $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$ is the set of all polynomials with convergent area in the interval* $[0, 1]$*, for which the discrete entries of $A_{\bar{i}}$ (for all fixed i) do not sum to zero. That is, for which $\sum_j a_{i,j} \neq 0$ (for which the net area is not zero). Let us agree to call this set* ***normalizable****.*
- *Let $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$ is the set of all equivalence classes that partition $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$, with equivalence class representatives are normalized elements according to* ***Claim 3.1****.*
- *Let $\mathbf{\Omega}(\mathbb{Z}^+, < 0)$ is the set for which $\sum_j a_{i,j} < 0$ (for which the net area is negative).*
- *Let $\mathbf{\Omega}(\mathbb{Z}^+, > 0)$ is the set for which $\sum_j a_{i,j} > 0$ (for which the net area is positive).*
- *Let $\mathbf{\Omega}(\mathbb{Z}^+, 1, \mathbb{R}^+ \cup \{0\})$ is the set for which the net area is $\sum_j a_{i,j} = 1$ and for which the polynomials it contains has image in $\mathbb{R}^+ \cup \{0\}$; that is, probability distributions (*strict Pasquali patches*) in the interval* $[0, 1]$.

It seems clear that the following important relations hold (there may be others, but we want to focus the object of our study to probability distributions):

$$\mathbf{\Omega}(\mathbb{Z}^+) \supset \mathbf{\Omega}(\mathbb{Z}^+, \neq 0) \supset \mathbf{\Omega}(\mathbb{Z}^+, > 0) \supset \mathbf{\Omega}(\mathbb{Z}^+, 1, \mathbb{R}^+ \cup \{0\})$$



Notice whatever we disprove for the subsets will be disproven for $\mathbf{\Omega}(\mathbb{Z}^+)$.

3.1. Equivalence classes of $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$: $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$.

Claim 3.1. ***(August 12, 2014)*** *Take an element* $\omega_i(x) \in \mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$. *Define a relation on* $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$ *by setting* $\omega_1(x) \sim \omega_2(x)$ *if*

$$\frac{A_1}{\sum_j a_{1,j}} = \frac{A_2}{\sum_j a_{2,j}}$$

This in fact defines an equivalence relation on the set $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$. *Observe the equality is ascertained entry-by-entry, since* A_1 *and* A_2 *are row vectors of matrix* A.

Proof The equivalence relation is defined by the rules (reflexivity, symmetry, transitivity) of equality. Thus we have:

- *Reflexivity.* Pick any $\omega(x) \in \mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$, and apply the definition of equivalence relation, so that
$$\frac{A_i}{\sum_j a_{i,j}} = \frac{A_i}{\sum_j a_{i,j}}$$
for any fixed i. Reflexivity of the equivalence relation follows from reflexivity of equality.
- *Symmetry.* Pick $(\omega_1(x), \omega_2(x)) \in \mathbf{\Omega}(\mathbb{Z}^+, \neq 0) \times \mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$ so that
$$\frac{A_1}{\sum_j a_{1,j}} = \frac{A_2}{\sum_j a_{2,j}}$$
or $\omega_1(x) \sim \omega_2(x)$. It is easy to see that by the symmetric property of equality,
$$\frac{A_2}{\sum_j a_{2,j}} = \frac{A_1}{\sum_j a_{1,j}}$$
or $\omega_2(x) \sim \omega_1(x)$
- *Transitivity.* Pick $(\omega_1(x), \omega_2(x)), (\omega_2(x), \omega_3(x)) \in \mathbf{\Omega}(\mathbb{Z}^+, \neq 0) \times \mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$, so that $\omega_1(x) \sim \omega_2(x)$ and $\omega_2(x) \sim \omega_3(x)$. This translates to
$$\frac{A_1}{\sum_j a_{1,j}} = \frac{A_2}{\sum_j a_{2,j}}$$
and
$$\frac{A_2}{\sum_j a_{2,j}} = \frac{A_3}{\sum_j a_{3,j}}$$

By transitivity of equality, it follows that

$$\frac{A_1}{\sum_j a_{1,j}} = \frac{A_3}{\sum_j a_{3,j}}$$

or $\omega_1(x) \sim \omega_3(x)$.

□

Example 3.1. ***(August 12, 2014)*** *Here we introduce notation for equivalent $\omega \in \mathbf{\Omega}(\mathbb{Z}^+)$. Suppose that in the $\bar{i}$th row of A, that is in $A_{\bar{i}}$, we have*

$$A_{\bar{i}} = \begin{bmatrix} 1 & 1 & 0 & \dots \end{bmatrix}$$

Let us agree that the notation

$$\omega_{\begin{bmatrix} 1 & 1 \end{bmatrix}}(x) = 1 + 2x$$

represents the polynomial generated by such. Then

$$\omega_{\begin{bmatrix} 1 & 1 \end{bmatrix}}(x) \sim \omega_{\begin{bmatrix} 2 & 2 \end{bmatrix}}(x) \sim \dots \sim \omega_{\begin{bmatrix} z & z \end{bmatrix}}(x)$$

with $z \in \mathbb{R} \setminus \{0\}$ fall into the equivalence class represented by

$$\omega^{\circ}_{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}}(x) = \frac{1}{2} + x$$

We shall call this last polynomial the lowest terms *polynomial, and it is easy to check that it in fact is in* lowest terms *by adding the entries between brackets. If the result is* 1*, it is. However we make explicit note of this by adding a circle at the top of the ending bracket (this can be especially useful when we have infinite polynomials, since this saves us the trouble of adding infinite sequences to ascertain equality to* 1*).*

Next let us assume that we have an infinite sequence, such as $\frac{1}{n^2}$. We can write such as:

$$\omega_{\begin{bmatrix} \frac{1}{n^2}\Big|_1^\infty \end{bmatrix}}(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

where the formula represents how we will write the entries, and the pipe represents the range of the index. We remark that this polynomial converges in area in the interval from $[0, 1]$ *(Basel problem), but particularly we would not represent polynomials in this way if we were not guaranteed convergence already of the sequence at the sum. Thus we emphasize that this notation is not for all polynomials. It is for polynomials that converge in area in the interval between* $[0, 1]$*: polynomials $\omega(x)$.*

Lastly, if the sequence were a finite sequence with many (several) entries, we could write:

$$\omega_{\begin{bmatrix} \frac{1}{n^2}\Big|_1^4 \end{bmatrix}}(x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4}$$

Definition 3.2. ***(September 22, 2014)*** *Define $\omega_{[1\cdots]}(x)$ which lies outside of $\mathbf{\Omega}(\mathbb{Z}^+)$ to be the* one *polynomial so that*

$$\omega_{[1\cdots]}(x) = 1 + 2 \cdot x + 3 \cdot x^2 + \cdots$$

Notice this is a separate object, in that it does not lie inside of any set $\mathbf{\Omega}$, since the sum of ones does not converge.

Definition 3.3. ***(August 13, 2014)*** *Define $\omega_{[0]}(x) \in \mathbf{\Omega}(\mathbb{Z}^+)$ to be the* zero *polynomial so that*

$$\omega_{[0]}(x) = \mathbf{0} = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots$$

Notice we cannot normalize the polynomial (we cannot divide by zero), so this element lies outside $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$. However it does lie in $\mathbf{\Omega}(\mathbb{Z}^+)$ because the sum of the zero entries converge (to zero).

3.2. Binary Operation $\oplus$ on $\mathbf{\Omega}(\mathbb{Z}^+)$.

Definition 3.4. ***(August 13, 2014)*** *Define the binary operator $\oplus$ on $\mathbf{\Omega}(\mathbb{Z}^+)$, so that $\oplus\colon \mathbf{\Omega}(\mathbb{Z}^+)\times\mathbf{\Omega}(\mathbb{Z}^+)\to\mathbf{\Omega}(\mathbb{Z}^+)$ takes two polynomials $\omega_1(x)$ and $\omega_2(x)$ and converts them to another via addition of coefficients term-by-term.*

Claim 3.2. ***(August 25, 2014)*** $\big(\mathbf{\Omega}(\mathbb{Z}^+),\oplus\big)$ *is an abelian group.*

- *Closure.* Take elements $\omega_1(x),\omega_2(x)\in\mathbf{\Omega}(\mathbb{Z}^+)$, which we can write explicitly as $\omega_{[\rho_1(n)]}(x)$ and $\omega_{[\rho_2(n)]}(x)$, and which have convergent area $\sum_n \rho_1(n)=\alpha_1$ and $\sum_n \rho_2(n)=\alpha_2$. Then
$$\omega_1(x)\oplus\omega_2(x)=\omega_{[\rho_1(n)]}(x)\oplus\omega_{[\rho_2(n)]}(x)=\omega_{[\rho_1(n)+\rho_2(n)]}(x)$$
has area $\sum_n(\rho_1(n)+\rho_2(n))$ or $\alpha_1+\alpha_2$. Since both $\alpha_1,\alpha_2\in\mathbb{R}$, their sum is in $\mathbb{R}$ by closure of the reals. We conclude $\omega_{[\rho_1(n)+\rho_2(n)]}(x)$ has convergent area, and belongs to $\mathbf{\Omega}(\mathbb{Z}^+)$, or $\omega_1(x)\oplus\omega_2(x)\in\mathbf{\Omega}(\mathbb{Z}^+)$.
- *Associativity.* Take elements $\omega_1(x),\omega_2(x),\omega_3(x)\in\mathbf{\Omega}(\mathbb{Z}^+)$. The fact that $(\omega_1(x)\oplus\omega_2(x))\oplus\omega_3(x)=\omega_1(x)\oplus(\omega_2(x)\oplus\omega_3(x))$ follows from associativity of the reals. Thus we have that
$$\big(\omega_{[\rho_1(n)]}(x)\oplus\omega_{[\rho_2(n)]}(x)\big)\oplus\omega_{[\rho_3(n)]}(x)=\big(\omega_{[\rho_1(n)+\rho_2(n)]}(x)\big)\oplus\omega_{[\rho_3(n)]}(x)=\omega_{[\rho_1(n)+\rho_2(n)+\rho_3(n)]}(x)$$
On the other hand,
$$\omega_{[\rho_1(n)]}(x)\oplus\big(\omega_{[\rho_2(n)]}(x)\oplus\omega_{[\rho_3(n)]}(x)\big)=\omega_{[\rho_1(n)]}(x)\oplus\big(\omega_{[\rho_2(n)+\rho_3(n)]}(x)\big)=\omega_{[\rho_1(n)+\rho_2(n)+\rho_3(n)]}(x)$$
- *Identity.* We show that $\omega_{[0]}(x)$ is an identity element, and that
$$\omega_1(x)\oplus\omega_{[0]}(x)=\omega_{[0]}(x)\oplus\omega_1(x)=\omega_1(x)$$
We have
$$\omega_{[\rho(n)]}(x)\oplus\omega_{[0]}(x)=\omega_{[\rho(n)+0]}(x)=\omega_{[\rho(n)]}(x)$$
On the other hand,
$$\omega_{[0]}(x)\oplus\omega_{[\rho(n)]}(x)=\omega_{[0+\rho(n)]}(x)=\omega_{[\rho(n)]}(x)$$
- *Inverses.* We just need to show that an element of $\mathbf{\Omega}(\mathbb{Z}^+)$ together with its inverse yields the identity as we defined it. So take $\omega_1(x),\omega_1^{-1}(x)\in\mathbf{\Omega}(\mathbb{Z}^+)$, and let $\omega_1(x)=\omega_{[\rho(n)]}(x)$. The inverse element is $\omega_1^{-1}(x)=\omega_{[-\rho(n)]}(x)$, since
$$\omega_1(x)\oplus\omega_1^{-1}(x)=\omega_{[\rho(n)]}(x)\oplus\omega_{[-\rho(n)]}(x)=\omega_{[\rho(n)-\rho(n)]}(x)=\omega_{[0]}(x)$$
A similar argument shows that
$$\omega_1^{-1}(x)\oplus\omega_1(x)=\omega_{[-\rho(n)]}(x)\oplus\omega_{[\rho(n)]}(x)=\omega_{[-\rho(n)+\rho(n)]}(x)=\omega_{[0]}(x)$$
- *Commutativity.* This follows from the fact that the summing operation is commutative. Thus
$$\omega_1(x)\oplus\omega_2(x)=\omega_{[\rho_1(n)]}(x)\oplus\omega_{[\rho_2(n)]}(x)=\omega_{[\rho_1(n)+\rho_2(n)]}(x)$$
By commutativity of $+$, we have
$$=\omega_{[\rho_2(n)+\rho_1(n)]}(x)=\omega_{[\rho_2(n)]}(x)\oplus\omega_{[\rho_1(n)]}(x)=\omega_2(x)\oplus\omega_1(x)$$

□

3.2.1. *Binary Operation $\oplus$ on $\mathbf{\Omega}(\mathbb{Z}^+,\neq 0)$.*

Claim 3.3. ***(August 25, 2014)*** $\big(\mathbf{\Omega}(\mathbb{Z}^+,\neq 0),\oplus\big)$ *is not closed.*

Proof by Counterexample Take $\omega_1(x)=\omega_{[\rho(n)]}(x)$ with $\sum_n\rho(n)\neq 0$ and $\omega_2(x)=\omega_{[-\rho(n)]}(x)$ with $-\sum_n\rho(n)\neq 0$, so that both $\omega_1(x),\omega_2(x)\in\mathbf{\Omega}(\mathbb{Z}^+,\neq 0)$. But then
$$\omega_1(x)\oplus\omega_2(x)=\omega_{[\rho(n)]}(x)+\omega_{[-\rho(n)]}(x)=\omega_{[\rho(n)-\rho(n)]}(x)=\omega_{[0]}(x)$$
The issue is that $\omega_{[0]}(x)\notin\mathbf{\Omega}(\mathbb{Z}^+,\neq 0)$, since $\sum_n 0=0$.

This is by no means the only counterexample. Take for example $\omega_{\left[\begin{array}{cc} 2 & 0\end{array}\right]}(x)$ and $\omega_{\left[\begin{array}{cc} 0 & -2\end{array}\right]}(x)$, which converge in area to $\alpha_1=2$ and $\alpha_2=-2$ respectively. Using $\oplus$, we have that
$$\omega_1(x)\oplus\omega_2(x)=\omega_{\left[\begin{array}{cc} 2 & -2\end{array}\right]}(x)$$
with total area $\alpha_{1+2}=0$. This element is not in $\mathbf{\Omega}(\mathbb{Z}^+,\neq 0)$. □

3.2.2. *Binary Operation $\oplus$ on the elements of the specific equivalence class $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}$.*

Claim 3.4. ***(August 29, 2016)*** *$\left(\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}, \oplus\right)$ is not closed.*

Proof First, take $\omega_1(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}$, that is, with $\sum_n \rho(n) \neq 0$, $\omega_{[\rho(n)]}(x) \sim \omega_{\left[\frac{\rho(n)}{\sum_n \rho(n)}\right]}(x)$. Next, observe that $\omega_2(x) = \omega_{[-\rho(n)]}(x)$ is in the same equivalence class, for

$$\omega_{[-\rho(n)]}(x) \sim \omega_{\left[\frac{-\rho(n)}{\sum_n(-\rho(n))}\right]}(x) = \omega_{\left[\frac{\cancel{-}\rho(n)}{\cancel{-}\sum_n \rho(n)}\right]}(x) = \omega_{\left[\frac{\rho(n)}{\sum_n \rho(n)}\right]}(x)$$

But now

$$\omega_1(x) \oplus \omega_2(x) = \omega_{[\rho(n)]}(x) \oplus \omega_{[-(\rho(n))]}(x) = \omega_{[\rho(n)-\rho(n)]}(x) = \omega_{[0]}(x) \notin \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}$$

□

3.2.3. *Binary Operation $\oplus$ on the elements of the specific equivalence class $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$.*

Claim 3.5. ***(August 29, 2016)*** *$\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$ is an abelian group.*

Proof The inclusion of $\omega_{[0]}(x)$ will be very lucky, as it automatically renders the members of a specific equivalence class into an abelian group.

- *Closure.* Pick the $\bar{i}$ equivalence class in the space $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$. Next pick an element in that equivalence class $\omega_1(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}$, say $\omega_{[\rho(n)]}(x)$. From the proof of **Claim 3.4**, we know that $\omega_2(x) = \omega_{[-\rho(n)]}(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}$, and that $\omega_1(x) \oplus \omega_2(x) = \omega_{[0]}(x)$. But $\omega_{[0]}(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$ and the particular objection of **Claim 3.4** is resolved.

  Next, again pick two elements $\omega_1(x), \omega_2(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}}$, like: $\omega_1(x) = \omega_{[\rho_1(n)]}(x)$ and $\omega_2(x) = \omega_{[\rho_2(n)]}(x)$, so that $\rho_1(n) + \rho_2(n) \neq 0$. Since they belong to the same equivalence class, $\frac{\rho_1(n)}{\sum_n \rho_1(n)} = \frac{\rho_2(n)}{\sum_n \rho_2(n)} \Rightarrow \rho_2 = \rho_1(n) \cdot r$. Then:

  $$\omega_{[\rho_1(n)]}(x) \oplus \omega_{[\rho_2(n)]}(x) = \omega_{[\rho_1(n)+\rho_2(n)]}(x) \sim \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{\sum_n \rho_1(n)+\rho_2(n)}\right]}(x) = \omega_{\left[\frac{\rho_1(n)\cdot\cancel{(1+r)}}{\sum_n \rho_1(n)\cdot\cancel{(1+r)}}\right]}(x)$$

  shows we indeed remain in the same equivalence class with $\oplus$.

  Pick $\omega_1(x) = \omega_{[0]}(x), \omega_2(x) = \omega_{[\rho(n)]}(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$. It is easy to see that $\omega_1(x) \oplus \omega_2(x) = \omega_{[0+\rho(n)]}(x) = \omega_{[\rho(n)]}(x)$ and we have not left the equivalence class $\bar{i}$ union the zero element, even if we commute the elements.

  Finally, observe that $\omega_1(x) = \omega_2(x) = \omega_{[0]}(x)$ and $\omega_1(x) \oplus \omega_2(x) = \omega_{[0+0]}(x) = \omega_{[0]}(x)$ which again is in $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$.
- *Associativity.* Associativity follows from Associativity of $+$. Observe that $\oplus$ translates to $+$ at the subscript. Picking $\omega_1(x), \omega_2(x), \omega_3(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$, we have

  $$\begin{aligned}\left(\omega_{[\rho_1(n)]}(x) \oplus \omega_{[\rho_2(n)]}(x)\right) \oplus \omega_{[\rho_3(n)]}(x) &= \omega_{[(\rho_1(n)+\rho_2(n))+\rho_3(n)]}(x) \\ &= \omega_{[\rho_1(n)+(\rho_2(n)+\rho_3(n))]}(x) \\ &= \omega_{[\rho_1(n)]}(x) \oplus \left(\omega_{[\rho_2(n)]}(x) \oplus \omega_{[\rho_3(n)]}(x)\right)\end{aligned}$$
- *Identity.* We show that $\omega_{[0]}(x)$ is an identity element.

  Let $\omega_1(x) = \omega_{[\rho(n)]}(x)$. Then

  $$\omega_{[\rho(n)]}(x) \oplus \omega_{[0]}(x) = \omega_{[\rho(n)+0]}(x) = \omega_{[\rho(n)]}(x)$$

  On the other hand,

  $$\omega_{[0]}(x) \oplus \omega_{[\rho(n)]}(x) = \omega_{[0+\rho(n)]}(x) = \omega_{[\rho(n)]}(x)$$
- *Inverses.* We just need to show that an element of $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$ together with its inverse yields the identity as we defined it. But we have done so already in the proof of **Claim 3.4**. An element $\omega_1(x) = \omega_{[\rho(n)]}(x)$ has inverse $\omega_2(x) = \omega_{[-\rho(n)]}(x)$. In the proof of **Claim 3.4** we have shown both $\omega_1(x), \omega_2(x) \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$. Thus

  $$\omega_{[\rho(n)]}(x) \oplus \omega_{[-\rho(n)]}(x) = \omega_{[\rho(n)-\rho(n)]}(x) = \omega_{[0]}(x)$$

  gives the identity element as we defined it, $\omega_{[0]}(x)$.

  Finally, observe that $\omega_{[0]}(x)$ is its own inverse, since $\omega_{[0]}(x) \oplus \omega_{[0]}(x) = \omega_{[0+0]}(x) = \omega_{[0]}(x)$

□

3.2.4. *Binary Operation $\oplus$ on equivalence class representatives $\omega^\circ(x) \in \big[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\big]$.* Equivalence class representatives have their own rules under $\oplus$.

Claim 3.6. ***(August 29, 2016)*** *Let $\omega_1^\circ(x), \omega_2^\circ(x) \in \big[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\big]$ are two equivalence class representatives in the space of equivalence classes $\big[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\big]$. Then $\omega^\circ_{[\rho(n)]}(x) \oplus \omega^\circ_{[\sigma(n)]}(x) \sim \omega^\circ_{\left[\frac{\rho(n)+\sigma(n)}{2}\right]}(x)$.*

Proof It is clear that with $\omega^\circ_{[\rho(n)]}(x)$ we have $\sum_n \rho(n) = 1$ and $\omega^\circ_{[\sigma(n)]}(x)$ implies $\sum_n \sigma(n) = 1$ (the superscript circle tells us so), they are two different equivalence class representatives (and therefore fill the slots of the row vector sufficiently differently to warrant different function representations, ρ and σ). Performing $\oplus$ yields:

$$\omega^\circ_{[\rho(n)]}(x) \oplus \omega^\circ_{[\sigma(n)]}(x) = \omega_{[\rho(n)+\sigma(n)]}(x) \sim \omega^\circ_{\left[\frac{\rho(n)+\sigma(n)}{\sum_n \rho(n)+\sigma(n)}\right]}(x) = \omega^\circ_{\left[\frac{\rho(n)+\sigma(n)}{\sum_n \rho(n)+\sum_n \sigma(n)}\right]}(x) = \omega^\circ_{\left[\frac{\rho(n)+\sigma(n)}{2}\right]}(x)$$

The neat thing is that $\oplus$ *averages* equivalence class representatives via similarity. □

We might do well to introduce the operator $\ddagger^2 \colon \mathbf{\Omega}(\mathbb{Z}^+) \times \mathbf{\Omega}(\mathbb{Z}^+) \to \mathbf{\Omega}(\mathbb{Z}^+)$, which averages two elements $\omega_1(x), \omega_2(x)$ in $\mathbf{\Omega}(\mathbb{Z}^+)$ in exactly the expected way:

$$\omega_{[\rho(n)]}(x) \ddagger^2 \omega_{[\sigma(n)]}(x) = \omega_{\left[\frac{\rho(n)+\sigma(n)}{2}\right]}(x)$$

Later, we will see the properties of the operator more in detail.

It is now easy to see the relationship between $\oplus$ on equivalence class representatives and $\ddagger^2$:

$$\begin{array}{ccccc}
\omega^\circ_{[\rho(n)]}(x) & \oplus & \omega^\circ_{[\sigma(n)]}(x) & \sim & \omega^\circ_{\left[\frac{\rho(n)+\sigma(n)}{2}\right]}(x) \\
 & & & & \Updownarrow \\
\omega^\circ_{[\rho(n)]}(x) & \ddagger^2 & \omega^\circ_{[\sigma(n)]}(x) & = & \omega^\circ_{\left[\frac{\rho(n)+\sigma(n)}{2}\right]}(x)
\end{array}$$

3.2.5. *Binary Operation $\oplus$ on $\mathbf{\Omega}(\mathbb{Z}^+, > 0)$ and $\mathbf{\Omega}(\mathbb{Z}^+, < 0)$.* We will show that $\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ and $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$ are at most closed and associative, and therefore at most semigroups.

Claim 3.7. ***(September 28, 2014)*** *$\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ and $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$ are closed and associative.*

Proof We have:

- *Closure* of $\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$. Take $\omega_{[\rho_1(n)]}(x)$ so that $\sum_n \rho_1(n) > 0$ and $\omega_{[\rho_2(n)]}(x)$ so that $\sum_n \rho_2(n) > 0$, both by definition of course belong in $\mathbf{\Omega}(\mathbb{Z}^+, > 0)$. Now the important thing to focus on here is the sum over n elements. We may take as axiomatic that the positive reals, $\mathbb{R}^+$, are closed. Therefore the sum of areas $\sum_n \rho_1(n) + \sum_n \rho_2(n)$ must be greater than zero, since both were a real number greater than zero in the first place. This of course implies that $\sum_n (\rho_1(n) + \rho_2(n)) > 0$ which is the definition of area of the element $\omega_{[\rho_1(n)+\rho_2(n)]}(x) = \omega_{[\rho_1(n)]}(x) \oplus \omega_{[\rho_2(n)]}(x)$. We can see now that $\omega_{[\rho_1(n)]}(x) \oplus \omega_{[\rho_2(n)]}(x) \in \mathbf{\Omega}(\mathbb{Z}^+, > 0)$.
- *Closure* of $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$. The argument for polynomials with net negative area is exactly the same, and stems from the fact that $\mathbb{R}^-$ is closed. We need only use the negative sign in our argument.
- *Associativity* of both $\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ and $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$. This follows from associativity of the reals, at the area level. So take elements $\omega_{[\rho_1(n)]}(x), \omega_{[\rho_2(n)]}(x), \omega_{[\rho_3(n)]}(x) \in \big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ or otherwise let the elements $\omega_{[\rho_1(n)]}(x), \omega_{[\rho_2(n)]}(x), \omega_{[\rho_3(n)]}(x) \in \big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$. It is certainly true that their convergent areas (*all* positive or negative) will satisfy

$$\left(\sum_n \rho_1(n) + \sum_n \rho_2(n)\right) + \sum_n \rho_3(n) = \sum_n \rho_1(n) + \left(\sum_n \rho_2(n) + \sum_n \rho_3(n)\right)$$

  because we are talking about sums that converge that map to a (positive or negative) real number. Translating to $\oplus$ notation at the polynomial level we have:

$$\big(\omega_{[\rho_1(n)]}(x) \oplus \omega_{[\rho_2(n)]}(x)\big) \oplus \omega_{[\rho_3(n)]}(x) = \omega_{[\rho_1(n)]}(x) \oplus \big(\omega_{[\rho_2(n)]}(x) \oplus \omega_{[\rho_3(n)]}(x)\big)$$

  By the way, notice we have proven associativity before, and we can argue that we inherit associativity from the set $\mathbf{\Omega}(\mathbb{Z}^+)$. In this particular approach, we argue that all the sums (areas) in the equation are either all positive or negative (no mixing is allowed or we would be outside the set of interest) and we still have associativity.

□

Claim 3.8. ***(September 29, 2014)*** *$\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ and $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$ have no identity element and no inverses under the operation $\oplus$.*

Proof We have that:

- *No identity.* Clearly the element $\omega_{[0]}(x)$ is neither in $\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ nor in $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$ because its area is exactly 0.
- *No inverses.* $\big(\mathbf{\Omega}(\mathbb{Z}^+, > 0), \oplus\big)$ has no inverses because such excludes the possibility of area elements that are negative, so that there is no way to obtain the zero polynomial with area 0. Conversely, $\big(\mathbf{\Omega}(\mathbb{Z}^+, < 0), \oplus\big)$ excludes the possibility of areas that are positive, and therefore one can never obtain the zero polynomial with area zero.

□

3.2.6. *Binary Operation $\oplus$ on $\big[\mathbf{\Omega}(\mathbb{Z}^+, > 0)\big]$.* We show that, for each equivalence class, the elements are closed and associative in that equivalence class under $\oplus$.

3.2.7. *Binary Operation $\oplus$ on $\mathbf{\Omega}(\mathbb{Z}^+, 1, \mathbb{R}^+ \cup \{0\})$.* We show that the set of probability distributions in the interval $[0, 1]$ are only associative with $\oplus$.

3.2.8. *Binary Operation $\oplus$ on $\big[\mathbf{\Omega}(\mathbb{Z}^+, 1, \mathbb{R}^+ \cup \{0\})\big]$.* We show that, for each equivalence class, the elements are closed and associative in that equivalence class under $\oplus$.

3.2.9. *Example: Binary Operation $\oplus$ on $\mathbf{\Omega}(\mathbb{Z}(2))$.*

3.3. Binary Operation $\odot$ on $\mathbf{\Omega}(\mathbb{Z}^+)$.

3.4. Binary Operation $\otimes$ on $\mathbf{\Omega}(\mathbb{Z}^+)$.

Definition 3.5. ***(September 30, 2014)*** *Define the binary operator $\otimes$ on $\mathbf{\Omega}(\mathbb{Z}^+)$, so that $\otimes\colon \mathbf{\Omega}(\mathbb{Z}^+) \times \mathbf{\Omega}(\mathbb{Z}^+) \to \mathbf{\Omega}(\mathbb{Z}^+)$ takes two polynomials $\omega_1(x)$ and $\omega_2(x)$ and converts them to another via multiplication of coefficients term-by-term.*

3.5. Binary Operation $\ddagger^2$ on $\mathbf{\Omega}(\mathbb{Z}^+)$.

Definition 3.6. ***(August 23, 2014)*** *Define the binary operator $\ddagger^2\colon \mathbf{\Omega}(\mathbb{Z}^+) \times \mathbf{\Omega}(\mathbb{Z}^+) \to \mathbf{\Omega}(\mathbb{Z}^+)$ in the following way. Take two elements $\omega_1(x), \omega_2(x) \in \mathbf{\Omega}(\mathbb{Z}^+)$. Then:*

$$\omega_1(x) \ddagger^2 \omega_2(x) = \omega_{[\rho_1(n)]}(x) \ddagger^2 \omega_{[\rho_2(n)]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x)$$

In other words, we take the average of each of the entries of A_i, A_j.

Claim 3.9. ***(August 25, 2014)*** *Each element in $\mathbf{\Omega}(\mathbb{Z}^+)$ is an identity to itself under the binary operation $\ddagger^2$.*

Proof Pick elements $\omega_1(x), \omega_2(x) \in \mathbf{\Omega}(\mathbb{Z}^+)$. We have to show that

$$\omega_1(x) \ddagger^2 \omega_2(x) = \omega_1(x) = \omega_2(x) \ddagger^2 \omega_1(x)$$

Thus it must be true from above that

$$\omega_{[\rho_1(n)]}(x) \ddagger^2 \omega_{[\rho_2(n)]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x) = \omega_{[\rho_1(n)]}(x)$$

But then we have that

$$\frac{\rho_1(n) + \rho_2(n)}{2} = \rho_1(n)$$

and $\rho_2(n) = \rho_1(n)$. Because of the commutativity of addition, the reverse argument is analogous. We are done. □

Claim 3.10. ***(August 25, 2014)*** *Each element in $\mathbf{\Omega}(\mathbb{Z}^+)$ is an inverse to itself under the binary operation $\ddagger^2$.*

Proof Pick an element $\omega_1(x), \omega_1^{-1}(x) \in \mathbf{\Omega}(\mathbb{Z}^+)$. We have to show that when we pair an element $\omega_1(x)$ with its inverse $\omega_1^{-1}(x)$ we obtain the identity: the original element by **Claim 3.9**.

$$\omega_1(x) \ddagger^2 \omega_1^{-1}(x) = \omega_1(x) = \omega_1^{-1}(x) \ddagger^2 \omega_1(x)$$

But notice this is the exact same argument as in **Claim 3.9**, since we must have that

$$\omega_{[\rho_1(n)]}(x) \ddagger^2 \omega_{[\rho_2(n)]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x) = \omega_{[\rho_1(n)]}(x)$$

We are done, we have proven this before. □

////// IN PROGRESS /////

3.5.1. *Binary Operation* $\ddagger^2$ *on* $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$.

Claim 3.11. ***(August 24, 2014)*** *Take any equivalence class* $[\omega^\circ(x)]_i \in \left[\mathbf{\Omega}^\star(\mathbb{Z}^+)\right]$. *These equivalence classes are closed under* $\ddagger^2$.

Proof Let, as before, $\omega_{[\rho_1(n)]}(x), \omega_{[\rho_2(n)]}(x)$ belong to an equivalence class $[\omega^\circ(x)]$. The boxed sum is:

$$\omega_{[\rho_1(n)]}(x) \ddagger^2 \omega_{[\rho_1(n)]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x)$$

Now, $\omega_{[\rho_1(n)]}(x) \sim \omega_{\left[\frac{\rho_1(n)}{\sum_n \rho_1(n)}\right]}(x)$ and $\omega_{[\rho_2(n)]}(x) \sim \omega_{\left[\frac{\rho_2(n)}{\sum_n \rho_2(n)}\right]}(x)$, and as before they belong to the same equivalence class and thus we again have $\rho_2(n) = r \cdot \rho_1(n)$ with r is the appropriate ratio of the sums. The boxed sum of the elements belongs to the equivalence class:

$$\omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x) \sim \omega_{\left[\frac{\frac{\rho_1(n)+\rho_2(n)}{2}}{\sum_n \frac{\rho_1(n)+\rho_2(n)}{2}}\right]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{\sum_n \rho_1(n)+\rho_2(n)}\right]}(x)$$

This exercise we have done before:

$$= \omega_{\left[\frac{\rho_1(n)+r\cdot\rho_1(n)}{\sum_n(\rho_1(n)+r\cdot\rho_1(n))}\right]}(x) = \omega_{\left[\frac{(r+1)\cdot\rho_1(n)}{(r+1)\sum_n \rho_1(n)}\right]}(x) = \omega_{\left[\frac{\rho_1(n)}{\sum_n \rho_1(n)}\right]}(x) \sim \omega^\circ(x)$$

□

Claim 3.12. ***(August 25, 2014)*** *The binary operation* $\ddagger^2$ *is not associative in* $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)$ *(and therefore not associative in* $\mathbf{\Omega}(\mathbb{Z}^+)$*).*

Proof We can argue from the specific to the more general: if it is not associative in the equivalence classes $[\omega^\circ(x)]_i \in \left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$, it is not associative anywhere in $\mathbf{\Omega}(\mathbb{Z}^+)$, since $\mathbf{\Omega}(\mathbb{Z}^+, \neq 0) \subset \mathbf{\Omega}(\mathbb{Z}^+)$. Pick any elements $\omega_1(x), \omega_2(x), \omega_3(x) \in [\omega^\circ(x)]_i$. We have on the one hand

$$\left(\omega_{[\rho_1(n)]}(x) \ddagger^2 \omega_{[\rho_2(n)]}(x)\right) \ddagger^2 \omega_{[\rho_3(n)]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x) \ddagger^2 \omega_{[\rho_3(n)]}(x) = \omega_{\left[\frac{\rho_1(n)+\rho_2(n)+2\rho_3(n)}{4}\right]}(x)$$

On the other hand

$$\omega_{[\rho_1(n)]}(x) \ddagger^2 \left(\omega_{[\rho_2(n)]}(x) \ddagger^2 \omega_{[\rho_3(n)]}(x)\right) = \omega_{[\rho_1(n)]}(x) \ddagger^2 \omega_{\left[\frac{\rho_2(n)+\rho_3(n)}{2}\right]}(x) = \omega_{\left[\frac{2\rho_1(n)+\rho_2(n)+\rho_3(n)}{4}\right]}(x)$$

□

3.6. Binary Operation $\ddagger^2$ on $[\omega^\circ(x)]_i$, Equivalence Class Representatives of $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$.

Definition 3.7. ***(August 23, 2014)*** *Define the binary operator* $\ddagger^2\colon \left[\mathbf{\Omega}^\star(\mathbb{Z}^+)\right] \times \left[\mathbf{\Omega}^\star(\mathbb{Z}^+)\right] \to \left[\mathbf{\Omega}^\star(\mathbb{Z}^+)\right]$ *in the following way. Take two elements* $[\omega^\circ(x)]_1, [\omega^\circ(x)]_2 \in \left[\mathbf{\Omega}^\star(\mathbb{Z}^+)\right]$. *Then:*

$$[\omega^\circ(x)]_1 \ddagger^2 [\omega^\circ(x)]_2 = \left[\omega^\circ_{[\rho_1(n)]}(x)\right] \ddagger^2 \left[\omega^\circ_{[\rho_2(n)]}(x)\right] = \left[\omega^\circ_{\left[\frac{\rho_1(n)+\rho_2(n)}{2}\right]}(x)\right]$$

Claim 3.13. ***(August 23, 2014)*** *Equivalence class representatives of* $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]$ *form a groupoid under the operation* $\ddagger^2$. *That is,* $\left(\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right], \ddagger^2\right)$ *is a group.*

Proof We need to show the following properties:

- *Closure.*

□

3.7. N-ary operation $\ddagger^n$.

3.8. N-ary operation $\ddagger^n_{\tau(m)}$.

3.9. Binary operation $\boxtimes^\downarrow$.

3.10. Binary operation $\boxtimes^\rightarrow$.

3.11. N-ary operation $\boxtimes^{\downarrow\rightarrow_a\downarrow\rightarrow_b\downarrow\rightarrow_c\cdots}$.

3.12. N-ary operation $\boxtimes^{\downarrow\rightarrow_a\downarrow\rightarrow_b\downarrow\rightarrow_c\cdots}_{\tau(m)}$.

3.13. Coefficient Extension: $\mathbf{\Omega}(\mathbb{R})$ and a New Class of Polynomial-like Functions.

4. Dynamics

4.1. Stability and Stationary States.

Corollary 4.1. ***(April 6, 2013)*** *If*

$$\lim_{t\to\infty} p_t(x,y) = p_\infty(x)$$

$t \in \mathbb{Z}^+$, *then*

$$\lim_{t\to\infty} \Delta P^t = 0$$

Proof Take

$$\Delta P^t = P^{t+1} - P^t$$

Applying the limit at infinity we get

$$\begin{aligned}
\lim_{t\to\infty} \Delta P^t &= \lim_{t\to\infty} \left[P^{t+1} - P^t\right] \\
&= \lim_{t\to\infty} P^{t+1} - \lim_{t\to\infty} P^t \\
&= P^\infty - P^\infty \\
&= 0
\end{aligned}$$

where the next-to-last step is justified by the hypothesis of the corollary (in other words, we assume convergence to $p_\infty(x)$). □

Corollary 4.2. ***(March 31, 2013)*** *If*

$$\lim_{t\to\infty} c_t(x) = p_\infty(x)$$

$t \in \mathbb{Z}^+$, *then*

$$\lim_{t\to\infty} \Delta c_t(x) = 0$$

Proof 1 From **Claim 2.28**,

$$c_{t+1}(x) = c_0(x) \star P^{t+1}$$

Thus we have that

$$\begin{aligned}
c_{t+1}(x) - c_t(x) &= c_0(x) \star P^{t+1} - c_0(x) \star P^t \\
&= c_0(x) \star \left(P^{t+1} - P^t\right)
\end{aligned}$$

where this part is justified by **Corollary 1.5, Distributive Property of the Star Operator**. Next at steady-state

$$\begin{aligned}
\lim_{t\to\infty} \Delta c_t(x) &= \lim_{t\to\infty} \left[c_0(x) \star \left(P^{t+1} - P^t\right)\right] \\
&= c_0(x) \star \left(\left[\lim_{t\to\infty} P^{t+1}\right] - \left[\lim_{t\to\infty} P^t\right]\right) \\
&= c_0(x) \star (P^\infty - P^\infty) \\
&= c_0(x) \star 0 \\
&= 0
\end{aligned}$$

Where we have pulled the limit under the star operator by using the hypothesis and conclusion of **Corollary 2.30**, **Corollary 4.1**, and we also used **Claim 1.6 Zero Property of the Star Product**. □

Proof 2 Using the **Claim 2.28** equivalence, we have

$$c_{t+1}(x) = c_t(x) \star P^1$$

and

$$c_{t+1}(x) - c_t(x) = (c_t(x) - c_{t-1}(x)) \star P^1$$

Taking the limit as $t \to \infty$ we get:

$$\begin{aligned}
\lim_{t\to\infty} \Delta c_t(x) &= \lim_{t\to\infty} (c_t(x) - c_{t-1}(x)) \star P^1 \\
&= \lim_{t\to\infty} \left(c_t(x) \star P^1 - c_{t-1}(x) \star P^1\right) \\
&= \lim_{t\to\infty} \left[c_t(x) \star P^1\right] - \lim_{t\to\infty} \left[c_{t-1} \star P^1\right] \\
&= \left[\lim_{t\to\infty} c_t(x)\right] \star P^1 - \left[\lim_{t\to\infty} c_{t-1}(x)\right] \star P^1 \\
&= c_\infty(x) \star P^1 - c_\infty(x) \star P^1 \\
&= (c_\infty(x) - c_\infty(x)) \star P^1 \\
&= 0 \star P^1 \\
&= 0
\end{aligned}$$

in which we used the hypothesis and limit implication of **Corollary 2.32**. Here we also used **Claim 1.6 Zero Property of the Star Product**. □

Construction 4.1. ***(March 31, 2013)*** *Let $k(c(x), t)$ is a piecewise continuous linear* functional *that is equal to $c_t(x)$ star-weighted by an arbitrary $g(y), g\colon [0, 1] \to \mathbb{R}$, in other words, $c_t(x) \star g(y)$. Create it so that between each unit time interval the slope is $\Delta c_t(x)$ for any x.*

Claim 4.3. ***(March 31, 2013)*** *If*

$$\lim_{t\to\infty} c_t(x) = p_\infty(x)$$

$t \in \mathbb{R}^+$, then

$$\lim_{t\to\infty} \frac{\partial k(c(x), t)}{\partial t} = 0$$

Proof For the intervals between $c_{t+1}(x)$ and $c_t(x)$, the derivative is $\Delta c_t(x)$ by construction. Thus we can take

$$\lim_{t\to\infty} \frac{\partial k(c(x), t)}{\partial t} = \lim_{t\to\infty} \Delta c_t(x) = 0$$

by **Claim 4.2**. Note that at infinity, convergence of $c_t(x)$ eventually smooths any kinks at integer time-steps and allows for the derivative to be defined there. □

Remark 4.1. ***(March 31, 2013)*** *The above claim intends to construct a situation in which probability on a* Pasquali patch *will accumulate or un-accumulate uniformly (linearly) in each unit of time interval, but stabilizing in the long-run,* for each x.

Remark 4.2. ***(March 31, 2013)*** *Anatomically, it seems clear that, because of its recursive dependence, $c_t(x)$ has basic form:*

$$c_t(x) = \overbrace{I(x)}^{\text{invariant in time}} + W(x) \cdot \overbrace{c_{t-1}(x) \star g(y)}^{\text{variable in time}}$$

The idea is to create a "continuous continuation" of $c_t(x)$ by using $k(c(x), t)$.

Claim 4.4. ***(March 31, 2013)*** *Take*

$$c_t^\circ(x) = \overbrace{I(x)}^{\text{invariant in time}} + W(x) \cdot \overbrace{k(c(x), t-1)}^{\text{variable in time}}$$

$t \in \mathbb{R}^+$, with the usual supposition of convergence to $p_\infty(x)$ of $c_t(x)$. Then the partial derivative

$$\lim_{t\to\infty} \frac{\partial c_t^\circ(x)}{\partial t} = 0$$

Proof By taking the partial derivative in the intervals where it is defined (except at positive integer t initially, but then for practical purposes the convergence smooths the derivative at these points too), we obtain

$$\begin{aligned}
\lim_{t\to\infty} \frac{\partial}{\partial t} \left[I(x) + W(x)k(x, t-1)\right] &= W(x) \lim_{t\to\infty} \frac{\partial}{\partial t} \left[k(c(x), t-1)\right] \\
&= W(x) \cdot 0 \\
&= 0
\end{aligned}$$

where we used **Claim 4.3**. □

Remark 4.3. *The newly created function $c_t^\circ(x)$ is now continuous in t and not just defined at integer time-steps. This in essence describes a manner in which to define in-between powers of* Pasquali patches*. Sometimes it is possible to find $k(x,t)$ explicitly, as the next chapter shows. However, although we may guarantee $c_t(x)$ is a probability distribution at $t \in \mathbb{Z}^+ \cup \{0\}$, the uniform accumulation of $k(c(x),t)$ at in-between times may not sum to 1 in the integral (due to the linear conversion to the next integer value of t). This may require a different definition of the accumulation (than linear). However at the limit of time at infinity this is unimportant. In addition, if time is quantized, the continuous description may even be superfluous.*

Nevertheless, the continuous description is important and in the next chapter(s) we show how we can obtain a *Pasqualian*: a function which describes the probability and propagation of probability states at every point in time, not just integer time-steps.

5. The Pasqualian

Definition 5.1. ***(August 4, 2014)*** *A* Pasqualian *is the function $p(x,y,t)$, continuous in all variables, which describes at each integer $t \in \mathbb{Z}^+$ the* Pasquali patch powers *of a system. In particular, $p(x,y,1) = p_1(x,y)$, the initial* Pasquali patch*, and $\lim_{t\to\infty} p(x,y,t) = p_\infty(x)$.*

Now, in addition, we seek a function which is a Pasquali patch *at non-integer times or (equivalently) at all times, and we therefore require in fact that $\int_0^1 p(x,y,t)\,dx = u(y) = 1$ be true $\forall y,t$ are continuous. The requirement that the* Pasqualian *describe a* Pasquali patch *at all points in time causes us to conclude that the range of t is ≥ 0; that is, $\mathbb{R}^+ \cup \{0\}$, although we will show we can extend the range to all of $\mathbb{R}$. The range of x,y is $\mathbb{R}$, except when we talk about a* strict Pasqualian*, in which case the range will be $\mathbb{R}^+ \cup \{0\}$ and the system will describe probability evolutions.*

Furthermore,

$$\lim_{t\to\infty} \frac{\partial p(x,y,t)}{\partial t} = 0$$

In other words, the value of the Paqualian *at infinity stabilizes smoothly.*

Example 5.1. ***(August 4, 2014)*** *Take for example the* Pasquali patch $p_1(x,y) = 1+\cos(2\pi x)\cos(2\pi y)$ *with powers:*

$$\begin{aligned}
p_1(x,y) &= 1 + \cos(2\pi x)\cos(2\pi y)\\
p_2(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2}\\
p_3(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{4}\\
p_4(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{8}\\
p_5(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{16}\\
p_6(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{32}\\
&\vdots\\
p_n(x,y) &= 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2^{(n-1)}}
\end{aligned}$$

It is clear that the Pasqualian *is $p(x,y,t) = 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2^{(t-1)}}$. Notice we could obtain the* Pasqualian *by an inductive argument, much like in* ***Example 2.2****, and assuming interpolation at non-integer t.*

Remark 5.1. ***(August 4, 2014)*** *Notice that the* Pasqualian *provides us with immense power to invert the star product transform. Assume the inverse of the generator* Pasquali patch *exists. If we look at $t = 2$, and the corresponding* Pasquali patch *$p_2(x,y) = p(x,y,2)$, for example, the application of the inverse generator* Pasquali patch *via the* Pasqualian *would yield $p(x,y,1)$. Applying the inverse again we obtain $p(x,y,0)$. If we keep applying such, all of the sudden we can keep inverting into the negative powers, by calculating $p(x,y,t)$ with $t \in \mathbb{Z}^-$, something we could not do before with the star product transform. Forward and backward star-product multiplications are translated to sums or subtractions in the variable t (kind of what the logarithm does for usual multiplications). Finally, we have not only the inverse of the generator, but any integer power, for we can always subtract or sum a number of steps to obtain this first* Pasquali patch.

Remark 5.2. ***(August 4, 2014)*** *Notice that the* Pasqualian *provides us with an additional advantage, and that is calculating* intermediate Pasquali patches*, or* Pasquali patches *for non-integer values of t. This is extraordinary in that the procedure essentially defines the process of taking roots on the star-product transform. Because of* ***Definition 5.1****, such intermediate powers or roots are guaranteed to be* Pasquali patches *as well.*

Claim 5.1. ***(August 4, 2014)*** *Not all systems have a* Pasqualian.

Proof by Counterexample Take **Example 2.2** which differs from **Example 5.1** by a negative sign. Recall we could, via induction, describe integer *Pasquali patches* by the equation

$$p_n(x, y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{(-2)^{(n-1)}}$$

However we cannot make n continuous because of the negative sign in front of the 2, and since the equation is therefore defined only at integer n, a *Pasqualian* does not exist. (We merely have a formula that describes the system at integer times). □

Remark 5.3. ***(August 4, 2014)*** *We emphasize the fact that a* Pasqualian *can be obtained as we did: via inductive examination, by looking at* Pasquali patch *sequences coming from a generator, and then assuming continuity in the step variable to interpolate (although this technique may not produce a* Pasqualian *for all systems, as we have seen in* ***Claim 5.1****). There are special systems with particular form which we will describe presently.*

5.1. Basic Form.

Remark 5.4. ***(August 11, 2014)*** *We may posit that the basic form of the* Pasqualian *is something like $p(x, y, t) = X(x)Y(y)T(t)$. However this specification is not adequate, since by* ***Definition 5.1*** *$\int_0^1 p(x, y, t)\,dx = 1$ should be a constant (equal to 1) and not a function of t. This simple observation causes us to amend the definition to something like $p(x, y, t) = M(x, y) \cdot T(t) + C(x, y)$, because then the function of t can vanish when we integrate across x, provided $M(x, y)$ integrates to zero and $C(x, y)$ to 1 as the definition requires. We work with this form of the* Pasqualian *and see what insights we can obtain.*

5.1.1. $p(x, y, t) = M(x, y) \cdot T(t) + C(x, y)$.
Properties of $T(t)$.

Claim 5.2. ***(August 10, 2014)***

$$\lim_{t\to\infty} T(t) = 0$$

Proof By **Definition 5.1** the following holds true:

$$\lim_{t\to\infty} p(x, y, t) = p_\infty(x)$$

Thus, we have that

$$\lim_{t\to\infty} (M(x, y) \cdot T(t) + C(x, y)) = M(x, y) \cdot \lim_{t\to\infty} T(t) + C(x, y) = p_\infty(x)$$

But then $p_\infty(x)$ is solely a function of x, so it follows that the time function must vanish at the limit. Thus we conclude $\lim_{t\to\infty} T(t) = 0$. We examine the function $C(x, y) = C(x) = p_\infty(x)$ more closely presently. □

Claim 5.3. ***(August 10, 2014)***

$$\lim_{t\to\infty} \frac{\partial T(t)}{\partial t} = 0$$

Proof Recall from **Definition 5.1** that

$$\lim_{t\to\infty} \frac{\partial p(x, y, t)}{\partial t} = 0$$

Thus we have that:

$$\lim_{t\to\infty} \frac{\partial\left[M(x, y) \cdot T(t) + C(x, y)\right]}{\partial t} = 0$$

or

$$M(x, y) \cdot \lim_{t\to\infty} \frac{\partial T(t)}{\partial t} = 0$$

For nonzero $M(x, y)$, the result follows. □

Properties of $M(x, y)$ and $C(x, y)$.

Claim 5.4. ***(June 4, 2014)*** *If $T(t)$ is never zero, then*

$$\int_0^1 M(x, y)\, dx = 0$$

Moreover

$$\int_0^1 C(x, y)\, dx = 1$$

Proof This follows from the fact that, at any time $t \in \mathbb{R}$ it must be true that the integral of the *Pasqualian* in the x direction must add up to one (at integer times the *Pasqualian* becomes a *Pasquali patch*, and according to the definition (**Definition 5.1**) such must integrate to 1 in the x direction). Thus we have that

$$\int_0^1 p(x, y, t)\, dx = \int_0^1 \left(M(x, y) \cdot T(t) + C(x, y)\right)\, dx = \int_0^1 M(x, y) \cdot T(t)\, dx + \int_0^1 C(x, y)\, dx = 1$$

The first integral, that is

$$\int_0^1 M(x, y) \cdot T(t)\, dx = 0$$

because the time function must vanish, otherwise we have the function add to the total area and such can never be constant or, in particular, 1 (it would have to have a varying time term). This implies

$$\int_0^1 M(x, y)\, dx = 0$$

provided $T(t)$ is never zero. But then it must be true that

$$\int_0^1 C(x, y)\, dx = 1$$

□

Corollary 5.5. ***(June 4, 2014)*** *$C(x, y)$ is a Pasquali patch.*

Proof Since

$$\int_0^1 C(x, y)\, dx = 1$$

by **Claim 5.4**, it follows from **Definition 2.1** that $C(x, y)$ is a Pasquali patch. □

Claim 5.6. ***(June 4, 2014)***

$$C(x, y) = C(x) = p_\infty(x)$$

Proof This follows from the fact that the *Pasqualian* must stabilize as we take the time limit to infinity. Thus

$$\lim_{t\to\infty} p(x, y, t) = \lim_{t\to\infty} [M(x, y) \cdot T(t) + C(x, y)] = C(x, y)$$

by taking the limit, but additionally the *Pasqualian* must stabilize to the stationary surface $p_\infty(x)$, and thus $C(x, y) = p_\infty(x)$ which we know is solely a function of x. □

Thus we have the new equation of probability evolution:

Remark 5.5. ***(August 11, 2014)*** *In summary, the* Pasqualian *can be written as*

$$\boxed{p(x, y, t) = M(x, y) \cdot T(t) + p_\infty(x)}$$

with the following properties:

(1) $\lim_{t\to\infty} T(t) = 0$
(2) $\lim_{t\to\infty} \frac{\partial T(t)}{\partial t} = 0$
(3) $T(t)$ *is never zero*
(4) $\int_0^1 M(x, y)\, dx = 0$
(5) $C(x, y) = p_\infty(x)$

*This shows that we can create an evolving (probability) system that is continuous in time by specifying the stationary surface, $M(x,y)$, and the manner by which we will decay to the stationary surface via $T(t)$. The stationary surface can be specified using **Claim 2.48**,* The Grand Classification Theorem. *Also, by using our **Construction 2.1** of* Pasquali patches, *we can exactly specify the form of such $M(x,y)$ by taking three different functions (so we have 3 degrees of freedom from 4 starting functions). Lastly, it seems clear that $T(t)$ will have to be of the family*

$$\mathbb{T} = \left\{a \cdot e^{-rt}, a \cdot t^{-n}, \dots\right\}$$

that are functions asymptotic to $y = 0$. We must just make sure that the subsequent Pasquali patches *that define the system are correctly specified at integer time.*

Form of M(x,y).

Claim 5.7. ***(June 4, 2014)*** *The specific form of $M(x,y)$, using **Construction 2.1**, for a* Pasqualian *$p(x,y,t)$ is*

$$M(x,y) = \left(f_1(x) - \frac{F_1}{F_2} f_2(x)\right) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)}$$

*where $F_1 = \int_0^1 f_1(x)\,dx$, $F_2 = \int_0^1 f_2(x)\,dx$ and B is given recursively by $B = p_\infty(x) \star g_1(y)$. Notice that we solely depend on the fact that B converges but that this will converge for arbitrary (analytic, polynomial, Taylor-expandable) choices $f_{1,2}(x)$ and $g_1(y)$ as shown in **Claim 2.25**.*

Proof Take the *Pasqualian* at time $t = 1$ so that

$$p(x,y,1) = M(x,y) \cdot T(1) + p_\infty(x)$$

Recall that the first Pasquali patch using **Construction 2.1** is

$$p_1(x,y) = f_1(x)g_1(y) + f_2(x)\frac{(1 - g_1(y)F_1)}{F_2}$$

and that the stationary patch is

$$p_\infty(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right) B$$

with $B = p_\infty(x) \star g_1(y)$ by **Claim 2.25**. Thus at time $t = 1$ we have the following equation:

$$p(x,y,1) = p_1(x,y)$$

which is

$$M(x,y) \cdot T(1) + p_\infty(x) = f_1(x)g_1(y) + f_2(x)\frac{(1 - g_1(y)F_1)}{F_2}$$

Next we solve for

$$M(x,y) \cdot T(1) = f_1(x)g_1(y) + \frac{f_2(x)}{F_2} - f_2(x)g_1(y)\frac{F_1}{F_2} - \frac{f_2(x)}{F_2} + f_2(x)\frac{F_1}{F_2}B - f_1(x)B$$

Simplifying we get

$$M(x,y) \cdot T(1) = (g_1(y) - B)\, f_1(x) + \frac{F_1}{F_2}\,(B - g_1(y))\, f_2(x)$$

and factoring we get

$$M(x,y) = \left(f_1(x) - \frac{F_1}{F_2} f_2(x)\right) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)}$$

as we wanted to show. □

Remark 5.6. ***(June 9, 2014)*** *For **Construction 2.1**, the equation of probability propagation (the* Pasqualian*) becomes:*

$$p(x,y,t) = \overbrace{\left[\left(f_1(x) - \frac{F_1}{F_2} f_2(x)\right) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)}\right] \cdot T(t)}^{M(x,y)\cdot T(t)} + \overbrace{\frac{f_2(x)}{F_2} - \left(\frac{f_2(x)F_1}{F_2} - f_1(x)\right) B}^{p_\infty(x)}$$

or put in another form:

$$p(x,y,t) = \left(f_1(x) - \frac{F_1}{F_2} f_2(x)\right) \cdot \left[g_1(y) \cdot \frac{T(t)}{T(1)} - B\left(\frac{T(t)}{T(1)} - 1\right)\right] + \frac{f_2(x)}{F_2}$$

Although we could arbitrarily choose $f_1(x), f_2(x)$ and $g_1(y)$, there is additional work to be done with the function $T(t)$, we must make sure to take Pasquali patch *powers (other than the generator) and match them to the* Pasqualian *to get the appropriate rate of decay. Skipping this step, however, makes for an excellent first approximation of the final form the* Pasqualian *will take.*

Proposition 5.8. ***(June 4, 2014)*** *In* ***Construction 2.1****, there is a particular case where we have the following special relation between functions of x:* $F_2 \cdot f_1(x) = F_1 \cdot f_2(x)$ *or* $f_2(x) = \alpha f_1(x)$ *with* $\alpha = \frac{F_2}{F_1}$.

Proof Since by **Claim 5.4** we have that $\int_0^1 M(x,y)\,dx = 0$, it follows that

$$\int_0^1 \left(f_1(x) - \frac{F_1}{F_2} f_2(x) \right) \cdot (g_1(y) - B)\; dx = 0$$

If the part that is a function of y is not zero, we must have

$$\int_0^1 \left(f_1(x) - \frac{F_1}{F_2} f_2(x) \right)\; dx = F_1 - \frac{F_1}{F_2} F_2 = 0$$

which we have shown that it is indeed the case for any function $f_1(x)$ and $f_2(x)$. Notice that there is a particular instance in which we may have the integral vanish if the part

$$\left(f_1(x) - \frac{F_1}{F_2} f_2(x) \right) = 0$$

before we integrate, in which case we have exactly the relationship $F_2 \cdot f_1(x) = F_1 \cdot f_2(x)$ or $f_2(x) = \alpha f_1(x)$ where α is the ratio between the areas of the two functions in the interval from 0 to 1, that is $\frac{F_2}{F_1}$. □

Corollary 5.9. ***(June 4, 2014)*** *For* ***Construction 2.1****,* $M(x,y) = 0$ *if and only if* $f_2(x) = \alpha f_1(x)$ *or* $g_1(y) = B$ *(or both).*

Proof We have:

- ⇒ Since
$$M(x,y) = \left(f_1(x) - \frac{f_2(x)}{\alpha} \right) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)} = 0$$
it follows that either $\left(f_1(x) - \frac{f_2(x)}{\alpha} \right) = 0$ in which case $f_2(x) = \alpha f_1(x)$; or $(g_1(y) - B) = 0$ in which case $g_1(y) = B$; or both situations apply.
- ⇐ On the other hand it can be easily seen from the formula that if $f_2(x) = \alpha f_1(x)$, then $\left(f_1(x) - \frac{f_2(x)}{\alpha} \right) = 0$ which causes $M(x,y) = 0$; if $g_1(y) = B$ then $(g_1(y) - B) = 0$ and $M = 0$; and if both situations apply then $M(x,y) = 0$ because $0 \cdot 0 = 0$.

□

Corollary 5.10. ***(June 4, 2014)*** *For* ***Construction 2.1****,* $M(x,y) = 0$ *if and only if the* Pasqualian *is* $p(x,y,t) = p_\infty(x)$ *(in other words, this situation describes the context in which the* Pasqualian *is* ***time-independent****).*

Proof We have:

- ⇒ The *Pasqualian* $p(x,y,t) = M(x,y) \cdot T(t) + p_\infty(x)$ becomes $p(x,y,t) = p_\infty(x)$ with the substitution $M(x,y) = 0$.
- ⇐ Since the *Pasqualian* is $p_\infty(x) = M(x,y) \cdot T(t) + p_\infty(x)$, it follows $M(x,y) \cdot T(t) = 0$. But $T(t)$ is never zero (see **Remark 5.5**) so it must be true that $M(x,y) = 0$.

□

Corollary 5.11. ***(June 4, 2014)*** *In* ***Construction 2.1****, the* Pasqualian $p(x,y,t) = p_\infty(x)$ *if and only if* $f_2(x) = \alpha f_1(x)$ *or* $g_1(y) = B$ *(or both).*

Proof This follows from **Corollary 5.9** and **Corollary 5.10**. □

Remark 5.7. *This works out nicely for us in the sense that, for* $f_2(x) = \alpha f_1(x)$*, the* Pasqualian *is independent of time (and the y variable). See* ***Remark 10.1*** *about time freezing and time-evolution. Thus imposing a single restriction on our choice of functions of x for* ***Construction 2.1*** *works out the* Pasqualian *to be a single function of x for all time (this concept we will refer to as* time-independence*).*

Corollary 5.12. ***(June 4, 2014)*** *For* ***Construction 2.1****, if* $f_2(x) = \alpha f_1(x)$*, then*

$$p(x,y,t) = p_\infty(x) = \frac{f_1(x)}{F_1} = \frac{f_1(x)}{\int_0^1 f_1(x)\,dx}$$

Proof Since for **Construction 2.1**,

$$p_\infty(x) = \frac{f_2(x)}{F_2} - \left(\frac{f_2(x)}{\alpha} - f_1(x)\right) B$$

and $f_2(x) = \alpha f_1(x)$, we have that

$$p_\infty(x) = \frac{\alpha f_1(x)}{F_2} - (0)\, B = \frac{f_1(x)}{F_1}$$

upon substitution. □

Example 5.2. ***(August 11, 2014)*** *Take **Example 5.1** with* Pasqualian

$$p(x, y, t) = 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2^{t-1}}$$

and $x, y \in [0, 1]$, $t \in \mathbb{R}$. Anatomically,

$$p(x, y, t) = \overbrace{\cos(2\pi x)\cos(2\pi y)}^{M(x,y)} \cdot \overbrace{\frac{1}{2^{t-1}}}^{T(t)} + \overbrace{1}^{p_\infty(x)}$$

*Notice how all functions conform to the specifications in **Remark 5.5**. In particular, let $f_1(x) = \cos(2\pi x)$, $g_1(y) = \cos(2\pi y)$, and $f_2(x) = 1$. Using **Remark 5.6**,*

$$p(x, y, t) = \left(f_1(x) - \frac{F_1}{F_2} f_2(x)\right) \cdot \left[g_1(y) \cdot \frac{T(t)}{T(1)} - B\left(\frac{T(t)}{T(1)} - 1\right)\right] + \frac{f_2(x)}{F_2}$$

matches exactly:

$$p(x, y, t) = \left(\cos(2\pi x) - \frac{0}{1} \cdot 1\right) \cdot \left[\cos(2\pi y) \cdot \frac{T(t)}{T(1)} - 0 \cdot \left(\frac{T(t)}{T(1)} - 1\right)\right] + \frac{1}{1}$$

where we are left only to determine $T(t)$ by matching Pasquali patches *using the star-product transform. Thus the above simplifies to*

$$p(x, y, t) = \cos(2\pi x)\cos(2\pi y) \cdot \frac{T(t)}{T(1)} + 1$$

We may know by performing a star-product transform on the original Pasquali patch *(that is, $p(x, y, 1)$) that*

$$p(x, y, 2) = \cos(2\pi x)\cos(2\pi y) \cdot \frac{T(2)}{T(1)} + 1 = \cos(2\pi x)\cos(2\pi y) \cdot \frac{1}{2} + 1$$

Thus we have that $T(2) = \frac{T(1)}{2}$. Knowledge that

$$p(x, y, 3) = \cos(2\pi x)\cos(2\pi y) \cdot \frac{1}{4} + 1$$

gives that $T(3) = \frac{T(1)}{4}$. Thus we deduce the relationship that $T(2) = 2 \cdot T(3)$. We may posit that $T(t) = a \cdot e^{-r \cdot t}$ for some $a, r \in \mathbb{R}$. Substituting we have $a \cdot e^{-2r} = 2 \cdot a \cdot e^{-3r}$ and $e^r = 2$ with $r = \ln(2)$. Thus $T(t)$ takes the form $a \cdot 2^{-t}$ and coefficient comparison (using $T(1)$) gives $a = 2$. Finally $T(t) = 2 \cdot 2^{-t}$ and

$$p(x, y, t) = \cos(2\pi x)\cos(2\pi y) \cdot \frac{1}{2^{t-1}} + 1$$

as we wanted to show, without inducing. We remark that we need knowledge of three Pasquali patches *since this allows us to create two equations for two unknowns, a and r.*

5.2. **General Basic Form.**

Remark 5.8. ***(August 11, 2014)*** *We may posit a more general basic form of the* Pasqualian *$p(x, y, t)$. Let $M_i, C\colon [0, 1]^2 \to \mathbb{R}$ and $T_i\colon \mathbb{R} \to \mathbb{R}$. We can define the* Pasqualian *by:*

$$p(x, y, t) = M_1(x, y) \cdot T_1(t) + M_2(x, y) \cdot T_2(t) + \ldots + M_k(x, y) \cdot T_k(t) + \ldots + C(x, y)$$

or, to put it differently

$$p(x, y, t) = \sum_{k=1}^{n} M_k(x, y) \cdot T_k(t) + C(x, y) = \mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + C(x, y)$$

Finally, because all claims we derived in the previous section apply, including all specifications for each of the functions, it follows that the general basic form is then

$$\boxed{p(x,y,t) = \mathbf{M}(\mathrm{x},\mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)}$$

with

(1) $\lim_{t\to\infty} \mathbf{T}(\mathrm{t}) = 0$
(2) $\lim_{t\to\infty} \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} = 0$
(3) $\mathbf{T}(\mathrm{t})$ *is never zero*
(4) $\int_0^1 \mathbf{M}(\mathrm{x},\mathrm{y})\, dx = 0$
(5) $C(x,y) = p_\infty(x)$

5.2.1. *Additional Claims.* These claims arise from the observation that it must be true that the *Pasqualian*, as an extension of *Pasquali patch* powers, conforms to the following by **Definition 5.1** (see also **Claim 1.10**):

$$p(x,y,n) \star p(x,y,m) = p(x,y,n+m)$$

with $n, m \in \mathbb{Z}^+$.

Claim 5.13. ***(September 21, 2015)*** *If* $\mathbf{M}(\mathrm{x},\mathrm{y})$ *is separable and* $\mathbf{M}(\mathrm{x},\mathrm{y}) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})$ *(as in **Claim 5.7** for **Construction 2.1**), then:*

(1) $\mathbf{X}(\mathrm{x}) \star \mathbf{Y}(\mathrm{y}) = 1$
(2) $\mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) = \mathbf{T}(\mathrm{n}+\mathrm{m}) | n, m \in \mathbb{Z}^+ \Rightarrow \mathbf{T}(\mathrm{t})$ *is an exponential function,* $t \in \mathbb{R}$
(3) $p_\infty(x) \star \mathbf{Y}(\mathrm{y}) = 0$

Proof We start with

- $p(x,y,n) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{n}) + p_\infty(x)$
- $p(x,y,m) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{m}) + p_\infty(x)$
- $p(x,y,n+m) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{n}+\mathrm{m}) + p_\infty(x)$
- $p(x,y,n) \star p(x,y,m) = p(x,y,n+m)$

Thus we have that

$$[\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{n}) + p_\infty(x)] \star [\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{m}) + p_\infty(x)] = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{n}+\mathrm{m}) + p_\infty(x)$$

On the LHS the star product distributes and we are left with four terms. The first term is:

$$\begin{aligned}
[\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{n})] \star [\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{m})] &= \mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) \, [\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \star \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})] \\
&= \mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) \, j \left(\int_0^1 \mathbf{X}(1-\mathrm{y}) \cdot \mathbf{Y}(\mathrm{z}) \cdot \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \, dy \right) \\
&= \mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) \, j \left(\mathbf{Y}(\mathrm{z}) \cdot \mathbf{X}(\mathrm{x}) \int_0^1 \mathbf{X}(1-\mathrm{y}) \mathbf{Y}(\mathrm{y}) \, dy \right) \\
&= \mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) \cdot \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \int_0^1 \mathbf{X}(1-\mathrm{y}) \mathbf{Y}(\mathrm{y}) \, dy \\
&= \mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) \cdot \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot A
\end{aligned}$$

with A is a constant and $A = \mathbf{X}(\mathrm{x}) \star \mathbf{Y}(\mathrm{y}) = 1$ and $\mathbf{T}(\mathrm{n}) \cdot \mathbf{T}(\mathrm{m}) = \mathbf{T}(\mathrm{n}+\mathrm{m})$. This first term is the one that leads to the conclusions of the claim. The notion that the time function is exponential will be further reinforced when we examine the derivative of the *Pasqualian*.

Next, notice that the cross terms from the star product must vanish. Thus it must hold true that:

$$[\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{n})] \star p_\infty(x) + p_\infty(x) \star [\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{m})] = 0$$

We will show both statements

$$\begin{aligned}
[\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})] \star p_\infty(x) &= 0 \\
p_\infty(x) \star [\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})] &= 0
\end{aligned}$$

are true. The first statement is easiest to prove, and the second one would follow from the above equation. Thus take

$$\begin{aligned}
[\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})] \star p_\infty(x) &= j \left(\int_0^1 \mathbf{X}(1-\mathrm{y}) \cdot \mathbf{Y}(\mathrm{z}) \cdot p_\infty(x) \, dy \right) \\
&= j \left(\mathbf{Y}(\mathrm{z}) \cdot p_\infty(x) \cdot \int_0^1 \mathbf{X}(1-\mathrm{y}) \, dy \right) \\
&= \mathbf{Y}(\mathrm{y}) \cdot p_\infty(x) \cdot \int_0^1 \mathbf{X}(1-\mathrm{y}) \, dy
\end{aligned}$$

Next, recall that, from **Remark 5.8**,

$$\begin{array}{rcl}\int_0^1 \mathbf{M}(\mathrm{x},\mathrm{y})\,dx = 0 & \Rightarrow & \int_0^1 \mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\,dx = \mathbf{Y}(\mathrm{y})\int_0^1 \mathbf{X}(\mathrm{x})\,dx = 0\\ & \Rightarrow & \int_0^1 \mathbf{X}(\mathrm{x})\,dx = 0\end{array}$$

Notice that a rotation of axis does not affect the integral, and

$$\int_0^1 \mathbf{X}(1-\mathrm{y})\,dy = 0$$

Therefore,

$$\begin{array}{rcl}[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})]\star p_\infty(x) & = & \mathbf{Y}(\mathrm{y})\cdot p_\infty(x)\cdot\int_0^1 \mathbf{X}(1-\mathrm{y})\,dy\\ & = & 0\end{array}$$

as we set out to show. Next, the equation $[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{n})]\star p_\infty(x)+p_\infty(x)\star[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{m})] = 0$ constrains the second statement $p_\infty(x)\star[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})] = 0$ to equal to zero as well. Conclusively,

$$p_\infty(x)\star\mathbf{Y}(\mathrm{y}) = 0$$

Finally, notice that the fourth and last term is $p_\infty(x)\star p_\infty(x)$ and is equal to $p_\infty(x)$ by comparison on the RHS and is true by, for example, **Claim 2.11**. □

Corollary 5.14. ***(September 22, 2015)*** *For* $\mathbf{M}(\mathrm{x},\mathrm{y}) = \mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})$*, either* $\mathbf{Y}(\mathrm{y}) = 0, \forall y$ *xor* $\mathbf{Y}(\mathrm{b_j}) < 0$ *for at least one* $b_j \in [0,1]$ *(it must have a region which is negative). Notice that the first case implies* $\mathbf{M}(\mathrm{x},\mathrm{y}) = 0$ *and* $p(x,y,t) = p(x) = p_\infty(x)$*.*

Proof From **Claim 5.13** we have that

$$p_\infty(x)\star\mathbf{Y}(\mathrm{y}) = 0$$

or

$$\int_0^1 p_\infty(1-y)\cdot\mathbf{Y}(\mathrm{y})\,dy = 0$$

Notice that in order for the integral to be zero, either $\mathbf{Y}(\mathrm{y}) = 0$, which in turn leads to

$$p(x,y,t) = \mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{t}) + p_\infty(x) = p_\infty(x)$$

xor, because $p_\infty(x)$ is a *Pasquali patch* and therefore $p_\infty(x) \geq 0 | x \in [0,1]$, in order for the integral to sum to zero there must be at least one b_j such that $\mathbf{Y}(\mathrm{b_j}) < 0$ and of magnitude enough to offset the accumulation of positive regions $\mathbf{Y}(\overline{\mathrm{b_j}})$ times the corresponding regions $p_\infty(1-\overline{b_j})$. □

Example 5.3. ***(February 10, 2016)*** *From* ***Claim 5.13*** *we have that* $\mathbf{X}(\mathrm{x})\star\mathbf{Y}(\mathrm{y}) = 1$ *and that* $p_\infty(x)\star\mathbf{Y}(\mathrm{y}) = 0$*, and from* ***Corollary 5.14*** *either* $\mathbf{Y}(\mathrm{y}) = 0 \Rightarrow p(x,y,t) = p_\infty(x), \forall x,y \in [0,1]$ *xor* $\mathbf{Y}(\mathrm{b_j}) < 0$ *for at least one* $b_j \in [0,1]$*. We show this is true for* ***Construction 2.1*** *with*

$$M(x,y) = \left(f_1(x) - \frac{F_1}{F_2}f_2(x)\right)\cdot(g_1(y)-B)\cdot\frac{1}{T(1)}$$

(see ***Claim 5.7****).*

For the claim that $\mathbf{X}(\mathrm{x})\star\mathbf{Y}(\mathrm{y}) = 1$*, it is evident we have to star the components of* $M(x,y)$*:*

$$\left(f_1(x) - \frac{F_1}{F_2}f_2(x)\right)\star(g_1(y)-B)\cdot\frac{1}{T(1)}$$

Now, if the star product of the claim is to be equal to 1 as ascertained, we must force the constant

$$T(1) = \left(f_1(x) - \frac{F_1}{F_2}f_2(x)\right)\star(g_1(y)-B)$$

We are done, since the time component is chosen independently to the x functions and the y functions.

To show that $p_\infty(x)\star\mathbf{Y}(\mathrm{y}) = 0$*, observe* $\mathbf{Y}(\mathrm{y}) = (g_1(y)-B)$ *say, and*

$$\begin{array}{rcl}p_\infty(x)\star(g_1(y)-B) & = & 0\\ p_\infty(x)\star g_1(y) - p_\infty(x)\star B & = & 0\\ p_\infty(x)\star g_1(y) & = & p_\infty(x)\star B\\ p_\infty(x)\star g_1(y) & = & B\end{array}$$

which is exactly the definition of B *(**Claim 2.25**). Next, say* $\mathbf{Y}(\mathrm{y}) = 0$ *and* $g_1(y) - B = 0$ *so that* $g_1(y) = B$*. Then* $M(x,y) = 0$ *and* ***Corollary 5.10*** *applies, with the implication being true. Finally, if* $g_1(b_j) - B < 0$ *for some collection* b_j *implies* $B > g_1(b_j)$*. Suppose otherwise, and* $B > g_1(y), \forall y$*. In particular,* $\int_0^1 B\,dy = B > \int_0^1 g_1(y)\,dy\,\forall y$*.*

Now let's take a closer look at the product $p(1-y)\cdot g_1(y)$ *and examine it piece by piece. We can confine* $0 \le p(1-y) \le 1$ *for any choice of* y*, for example, by requiring* $p(1-y) = 1\,\forall y$ *(a perfectly acceptable limiting* Pasquali patch*). It now follows that* $0 \le p(1-y)\cdot g_1(y) \le g_1(y)$ *for any choice of* y when $g_1(y)$ is 0 or positive*. Then*

$$0 \le \int_0^1 p(1-y)\cdot g_1(y)\,dy = p_\infty(x) \star g_1(y) \le \int_0^1 g_1(y)\,dy$$

when $g_1(y)$ is 0 or positive*. The middle part is exactly the definition of* B *from above, and*

$$0 \le B \le \int_0^1 g_1(y)\,dy$$

when $g_1(y)$ is 0 or positive*. This is a contradicting example that should suffice to prove our supposition wrong.*

Claim 5.15. ***(September 25, 2015)*** *Let* $\mathbf{M}(\mathrm{x},\mathrm{y}) = \mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})$*. Define* $r \in (0,1)$*, and* $n \in \mathbb{Z}^+$*. Then*

$$p(x,y,n) \star p(x,y,r) = p(x,y,n+r)$$

Proof The structure of this proof follows much of the same structure as **Claim 5.13**. From the LHS, we have

$$\begin{array}{rcl}[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{n})+p_\infty(x)]\star[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{r})+p_\infty(x)] & = & [\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{n})]\star[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{r})]+\\ & + & [\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{n})]\star p_\infty(x)+\\ & + & p_\infty(x)\star[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{r})]+\\ & + & p_\infty(x)\star p_\infty(x)\end{array}$$

From **Claim 5.13**, we have that:

$$\begin{array}{rcl}[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{n})+p_\infty(x)]\star[\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})\cdot\mathbf{T}(\mathrm{r})+p_\infty(x)] & = & \mathbf{T}(\mathrm{n})\cdot\mathbf{T}(\mathrm{r})\cdot\mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})+\\ & + & 0+\\ & + & 0+\\ & + & p_\infty(x)\end{array}$$

which is exactly $p(x,y,n+r)$. □

Corollary 5.16. ***(September 25, 2015)*** *Let* $\mathbf{M}(\mathrm{x},\mathrm{y}) = \mathbf{X}(\mathrm{x})\cdot\mathbf{Y}(\mathrm{y})$*. By* ***Claim 5.13*** *and* ***Claim 5.15****, the statement*

$$p(x,y,n) \star p(x,y,r) = p(x,y,n+r)$$

is true for $n \in \mathbb{Z}^+$ *and* $r \in (0,1)$*. In other words,* $\mathbf{T}(\mathrm{t})$ *is exponential for* $t \in [1,\infty)$*. We show that the statement is true for any real number by introducing a shift,* $s \in \mathbb{R}$*.*

Proof We have a *Pasqualian* $p(x,y,n+r)$ with $n+r \in [1,\infty)$. Let $s \in \mathbb{Z}$. Observe that the *Pasqualian*

$$p(x,y,n+r-s+s) = p(x,y,n+r) = p(x,y,n) \star p(x,y,r)$$

Thus

$$p(x,y,n+r-s+s) = p(x,y,n-s+r+s) = p(x,y,n-s) \star p(x,y,r+s)$$

Now the expression $p(x,y,r+s)$ exists because it consists of an integer and $r \in (0,1)$. In fact, the expression is equivalent to $p(x,y,s)\star p(x,y,r)$ (**Claim 5.15**). On the other hand, $p(x,y,n-s)$ is certainly a valid statement for $s \in \mathbb{Z}^- \cup \{0\}$, as we know from *Pasquali patch* powers, but it must also be valid for $s \in \mathbb{Z}^+$ in order for the initial expression to be correct. *We have just shown that we can extend the star product property to all integers.*

From here, we may either revisit **Claim 5.15** with $n \in \mathbb{Z}$ instead of $\mathbb{Z}^+$ or use the following argument. Take $s \in (0,1)$ and $r > s \Rightarrow r-s > 0$. Then

$$p(x,y,n+r+s-s) = p(x,y,n+s+r-s) = p(x,y,n+s) \star p(x,y,r-s)$$

The expression $p(x,y,n+s)$ exists by **Claim 5.15**, and $p(x,y,r-s)$ implies the existence of the *Pasqualian* in the interval $(0,1)$. With $r < s \Rightarrow r-s < 0$ we have existence in the interval $-(0,1)$. Finally, varying the value of $n \in \mathbb{Z}^-$ implies definition of the *Pasqualian* on the whole real line. *We have just shown that we can extend the star product property to all real numbers.* □

Example 5.4. ***(March 9, 2016)*** *We can use what we have learned to revisit the* Pasqualian *of some worked out examples. Take* ***Examples 5.1*** *and* ***5.2****:*

$$p(x,y,t) = 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2^{t-1}}$$

and $x, y \in [0,1], t \in \mathbb{R}$. *By* ***Corollary 5.16****, the range of* t *is now fully justified to be in* $\mathbb{R}$. *Next, observe we can define the functions* $f_1(x) = \cos(2\pi x), g_1(y) = \cos(2\pi y), f_2(x) = 1$, *and in particular notice* $\int_0^1 f_1(x)\,dx = \int_0^1 \cos(2\pi x)\,dx = 0$ *(recall* ***Construction 2.3****) so that*

$$M(x,y) = \left(f_1(x) - \frac{F_1}{F_2} f_2(x)\right) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)} = f_1(x) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)}$$

(from ***Claim 5.7****).* B *must be zero by anatomical comparison to the* Pasqualian*, but we can calculate it explicitly:*

$$B = p_\infty(x) \star g_1(y) = 1 \star g_1(y) = \int_0^1 g_1(y)\,dy = \int_0^1 \cos(2\pi y)\,dy = 0$$

which requires foreknowledge of the limiting surface $p_\infty(x) = 1$, *which again is easy to obtain via*

$$p(x,y,\infty) = \lim_{t\to\infty} p(x,y,t) = \lim_{t\to\infty} \left(1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2^{t-1}}\right) = 1$$

We remark there is an alternative way to calculate B *in the case* $p_\infty(x)$ *is not explicitly known using the infinite sum (integration by parts) definition of* B *(see* ***Example 2.5****). Thus we have*

$$M(x,y) = f_1(x) \cdot g_1(y) \cdot \frac{1}{T(1)}$$

and

$$T(1) = \cos(2\pi x) \star \cos(2\pi y) = \int_0^1 \cos\left(2\pi\left(1-y\right)\right) \cdot \cos(2\pi y)\,dy = \frac{1}{2}$$

Finally, the (canonical) definition of $M(x,y)$*:*

$$M(x,y) = 2 \cdot \cos(2\pi x) \cdot \cos(2\pi y)$$

implies the canonical form *of the* (strict) Pasqualian

$$p(x,y,t) = 2\cos(2\pi x)\cos(2\pi y) \cdot 2^{-t} + 1$$

with $x, y \in [0,1], t \in \mathbb{R}$

5.2.2. *Propagation of Zeros (or, Shadow Casting).* We can use the additional claims to show that zeros propagate along the time path of a *Pasqualian*.

Claim 5.17. ***(February 21, 2016)*** *Take a* Pasqualian $p(x,y,t)$ *so that* $p(a_j, y, 1) = 0$ *for all elements* a_j *in any open, disjoint sets* $A_j \subset [0,1]$ *(measure zero sets, e.g.). Then* $p(a_j, y, t) = 0\, \forall t \in \mathbb{R}$ *(the shadow is cast for all time). We may replace the value* 1 *with any value* $u \in \mathbb{R}$ *and obtain the same result. Also* $p(a_j, y, \infty) = 0$.

Proof Take a shift $s \in \mathbb{R}$. We can arrive at any time $t \in \mathbb{R}$ by using the shift s, as by defining $t = 1 + s$. So now look at $p(x,y,1+s)$. By **Corollary 5.16**,

$$p(x,y,1+s) = p(x,y,1) \star p(x,y,s)$$

It is clear that, evaluating at $x = a_j$, the product becomes:

$$p(a_j,y,1+s) = p(a_j,y,1) \star p(a_j,y,s) = 0 \star p(a_j,y,s) = 0$$

using **Claim 1.6**, and we are done. The argument works for any value $u \in \mathbb{R}$ instead of 1. Finally, recall $p(x,y,\infty) = \lim_{t\to\infty} p(x,y,t)$. Then:

$$p(a_j,y,\infty) = \lim_{t\to\infty} p(a_j,y,t) = \lim_{t\to\infty} 0 = 0$$

□

Claim 5.18. ***(February 21, 2016)*** *Take a* Pasqualian $p(x,y,t)$ *so that* $p(a_j,y,t) = 0$ *for all elements* a_j *in any open, disjoint sets* $A_j \subset [0,1]$ *for all* j*, as in* ***Claim 5.17****. Pick* $c_0(x)$ *and define* $c_t(x) = c_0(x) \star p(x,y,t)$, $t > 0$. *Then*

$$c_t(a_j) = 0$$

for such t *(notice* $c_0(a_j)$ *may not be equal to zero). In particular* $c_\infty(a_j) = 0$. *Note that we must define* $c_t(x)$ *for* $t > 0$ *because we observe, measure or conjecture the function* $c_0(x)$ *at a zeroth point in time – and otherwise we would obtain a contradiction that* $c_0(a_j)$ *may be non-zero.*

Proof We have:

$$c_t(x) = c_0(x) \star p(x, y, t)$$

and in particular

$$c_t(a_j) = c_0(a_j) \star p(a_j, y, t) = c_0(a_j) \star 0 = 0$$

with $t > 0$ by **Claim 1.6**. Next recall that $c_\infty(x) = \lim_{t\to\infty} c_t(x)$, and

$$c_\infty(a_j) = \lim_{t\to\infty} c_t(a_j) = \lim_{t\to\infty} 0 = 0$$

which is within the restriction $t > 0$ of t. □

5.3. **Other Forms.** The following example suggests that the General Basic Form of the *Pasqualian* may not be the only form for *Pasquali patch* and *strict Pasquali patch* evolutions.

Example 5.5. ***(March 18, 2016)*** *Recall* ***Example 2.1*** *with collection of* Pasquali patches

$$\begin{array}{rcl}
p(x,y) = x^2y^3 + 2x\left(1 - \frac{y^3}{3}\right) & = & 2x - \frac{2y^3x}{3} + y^3x^2 \\
p_2(x,y) & = & \frac{29x}{15} + \frac{y^3x}{90} + \frac{x^2}{10} - \frac{y^3x^2}{60} \\
p_3(x,y) & = & \frac{1741x}{900} - \frac{y^3x}{5400} + \frac{59x^2}{600} + \frac{y^3x^2}{3600} \\
\vdots & & \vdots \\
p_\infty(x) & = & \frac{2x^2}{21} + \frac{122x}{63}
\end{array}$$

and specification $f_1(x) = x^2, f_2(x) = 2x, g_1(y) = y^3, B = \frac{6}{61}$ *(see* ***Example 2.4****). In order to understand the time-dependent portion, we must subtract the invariant part. Thus we have:*

$$\begin{array}{rcccl}
p(x,y) - p_\infty(x) & = & \frac{4x}{63} - \frac{2y^3x}{3} - \frac{2x^2}{21} + y^3x^2 & = & \frac{x(3x-2)(7y^3-2/3)}{3^3\cdot 7} \\
p_2(x,y) - p_\infty(x) & = & -\frac{x}{315} + \frac{y^3x}{90} + \frac{x^2}{210} - \frac{y^3x^2}{60} & = & -\frac{x(3x-2)(7y^3-2)}{2^2\cdot 3^2\cdot 5\cdot 7} \\
p_3(x,y) - p_\infty(x) & = & -\frac{13x}{6300} - \frac{y^3x}{5400} + \frac{13x^2}{4200} + \frac{y^3x^2}{3600} & = & \frac{x(3x-2)(7y^3+78)}{2^4\cdot 3^3\cdot 5^2\cdot 7} \\
\vdots & & \vdots & & \vdots
\end{array}$$

Again, dividing out the invariant parts ($X(x)$ function part of separable $M(x,y)$) to isolate the time-varying portion:

$$\begin{array}{rcl}
Y(y)\cdot T(y,1) & = & \frac{7y^3-2/3}{3} \\
Y(y)\cdot T(y,2) & = & \frac{-(7y^3-2)}{2^2\cdot 5} \\
Y(y)\cdot T(y,3) & = & \frac{7y^3+78}{2^4\cdot 3\cdot 5^2} \\
\vdots & & \vdots
\end{array}$$

The example illustrates the fact that the Pasqualian *for this particular example does not exist (the alternating negative sign prohibits non-integer specifications of* Pasquali patches*, e.g.), but, in particular, it illustrates the possibility of (should a* Pasqualian *exist) having different regions of the* Pasquali patches *evolve toward steady-state at different rates; this is, having a time evolution which depends on x, y and not just t.*

Claim 5.19. ***(March 18, 2016)*** *Let $p(x,y,t) = M(x,y)\cdot T(x,y,t) + p_\infty(x)$ (not a General Basic Form of the* Pasqualian*). Next suppose $M(x,y) = M_x(x)\cdot M_y(y)$ is separable. Then $T(x,y,t)$ cannot be separable into distinct functions $T(x,y,t) = T_x(x)\cdot T_y(y)\cdot T_t(t)$.*

Proof For, suppose $T(x,y,t) = T_x(x)\cdot T_y(y)\cdot T_t(t)$. But then

$$p(x,y,t) = M_x(x)\cdot M_y(y)\cdot T_x(x)\cdot T_y(y)\cdot T_t(t) + p_\infty(x)$$

can be redefined as

$$p(x,y,t) = X(x)\cdot Y(y)\cdot T_t(t) + p_\infty(x)$$

which is a Basic Form of the *Pasqualian*. □

6. Consistency of $c_t(x)$

Remark 6.1. ***(December 28, 2014)*** *The time evolution of $c_t(x)$ is consistent with our observations using a* Pasqualian *of form*

$$p(x, y, t) = \mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)$$

In particular, notice that $c_t(x)$, using the star product, would have to be

$$c_t(x) = c_0(x) \star p(x, y, t)$$

*by **Claim 2.28**. This is,*

$$\begin{array}{rcl} c_t(x) & = & \int_0^1 c_0(x) \cdot p(x, y, t)\, dy \\ & = & \int_0^1 c_0(1-y) \cdot [\mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)]\ dy \\ & = & \int_0^1 c_0(1-y) \cdot [\mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t})]\ dy + \int_0^1 c_0(1-y) \cdot p_\infty(x)\, dy \\ & = & \mathbf{T}(\mathrm{t}) \cdot \int_0^1 c_0(1-y) \cdot \mathbf{M}(\mathrm{x}, \mathrm{y})\, dy + p_\infty(x) \int_0^1 c_0(1-y)\, dy \end{array}$$

Notice that $\int_0^1 c_0(1-y)\, dy = 1$ because $c_0(x)$ was defined a probability distribution in the interval $[0, 1]$, and integrating on a rotated axis still produces an area of 1. Thus we are left with

$$c_t(x) = \mathbf{T}(\mathrm{t}) \cdot \int_0^1 c_0(1-y) \cdot \mathbf{M}(\mathrm{x}, \mathrm{y})\, dy + p_\infty(x)$$

*If $\mathbf{M}(\mathrm{x}, \mathrm{y})$ is separable and $\mathbf{M}(\mathrm{x}, \mathrm{y}) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})$ (as in **Claim 5.7** for **Construction 2.1**), we have*

$$c_t(x) = \overbrace{p_\infty(x)}^{\textit{invariant in time}} + \overbrace{\mathbf{T}(\mathrm{t})}^{\textit{variable in time}} \cdot \overbrace{\mathbf{X}(\mathrm{x})}^{\textit{functions of } x} \cdot \overbrace{\int_0^1 c_0(1-y) \cdot \mathbf{Y}(\mathrm{y})\, dy}^{\textit{array of constants}}$$

*in fact consistent with our observation of $c_t^\circ(x)$ in **Remark 4.2**.*

Remark 6.2. ***(December 28, 2014)*** *In fact the consistency is extended when we consider the time evolution of $c_t(x)$. Thus:*

$$\begin{array}{rcl} \frac{\partial c_t(x)}{\partial t} & = & \frac{\partial}{\partial t}\left[p_\infty(x) + \mathbf{A} \cdot \mathbf{T}(\mathrm{t}) \cdot \mathbf{X}(\mathrm{x})\right] \\ & = & \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} \end{array}$$

and we are assured stability as $t \to \infty$:

$$\begin{array}{rcl} \lim_{t\to\infty} \frac{\partial c_t(x)}{\partial t} & = & \lim_{t\to\infty} \left[\mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}\right] \\ & = & \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \left[\lim_{t\to\infty} \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}\right] \\ & = & \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot [\mathbf{0}] \\ & = & 0 \end{array}$$

*by **Claim 5.2**, thereby verifying **Claim 4.4**.*

Claim 6.1. ***(December 30, 2014)*** *The derivative of the* Pasqualian $\frac{\partial p(x,y,t)}{\partial t}$ *when*

$$p(x, y, t) = \mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)$$

is

$$\frac{\partial p(x, y, t)}{\partial t} = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

Proof We are looking at a special case of a *Pasqualian* of form

$$p(x, y, t) = \mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)$$

Differentiating in the usual manner we get

$$\begin{array}{rcl} \frac{\partial p(x,y,t)}{\partial t} & = & \frac{\partial}{\partial t}\left(\mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)\right) \\ & = & \mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} \end{array}$$

With $\mathbf{M}(\mathrm{x}, \mathrm{y}) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})$ is separable we get

$$\frac{\partial p(x, y, t)}{\partial t} = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

□

Claim 6.2. ***(December 30, 2014)*** *If the* Pasqualian *has form*

$$p(x,y,t) = \mathbf{M}(\mathrm{x},\mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)$$

with $\mathbf{M}(\mathrm{x},\mathrm{y}) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y})$ *is separable, then*

$$\frac{\partial c_t(x)}{\partial t} = c_0(x) \star \frac{\partial p(x,y,t)}{\partial t}$$

Proof This is easily seen at the derivative. Thus take

$$\frac{\partial c_t(x)}{\partial t} = \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

as in **Remark 6.2**. Next consider the expression

$$c_0(x) \star \frac{\partial p(x,y,t)}{\partial t} = \int_0^1 c_0(1-y) \cdot \frac{\partial p(x,y,t)}{\partial t}\, dy$$

We have already calculated $\frac{\partial p(x,y,t)}{\partial t}$ in **Claim 6.1**, so we have

$$\begin{aligned} &= \int_0^1 c_0(1-y) \cdot \left(\mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \tfrac{\partial \mathbf{T}(\mathrm{t})}{\partial t}\right) dy \\ &= \mathbf{X}(\mathrm{x}) \cdot \tfrac{\partial \mathbf{T}(\mathrm{t})}{\partial t} \cdot \int_0^1 c_0(1-y) \cdot \mathbf{Y}(\mathrm{y})\, dy \\ &= \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \tfrac{\partial T(t)}{\partial t} \end{aligned}$$

Thus we have that

$$\frac{\partial c_t(x)}{\partial t} = c_0(x) \star \frac{\partial p(x,y,t)}{\partial t}$$

which is consistent with expression from **Proof 1** of **Claim 4.2**, where we had:

$$c_{t+1}(x) - c_t(x) = c_0(x) \star \left(P^{t+1} - P^t\right)$$

or equivalently

$$\Delta c_t(x) = c_0(x) \star \Delta P^t$$

as a continuous time analogue. Also notice that the result just proved implies we can differentiate under the integral (and across the star product), so that the integral is in fact absolutely convergent over all values of y:

$$\frac{\partial c_t(x)}{\partial t} = \frac{\partial}{\partial t}\left[c_0(x) \star p(x,y,t)\right] = c_0(x) \star \frac{\partial p(x,y,t)}{\partial t}$$

□

7. Quantum Mechanics / IN PROGRESS

7.1. The wavefunction.

Claim 7.1. ***(December 30, 2014)*** *Let* $p(x,y,t) = \mathbf{X}(\mathrm{x}) \cdot \mathbf{Y}(\mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)$. *Then*

$$\frac{\partial c_t(x)}{\partial t} \not\propto c_t(x)$$

as in the Shrödinger picture, but

$$\frac{\partial c_t(x)}{\partial t} \propto c_t(x) + q(x)$$

(the Pasquali picture). Furthermore,

$$\frac{\partial c_t(x)}{\partial t} \propto c_t(x) + q(x) \Rightarrow \mathbf{T}(\mathrm{t}) = a \cdot e^{-rt}$$

Proof We have: $\frac{\partial c_t(x)}{\partial t} \propto c_t(x)$ implies that the derivative is, say, star-weighted by a function $b(x)$, so that

$$\frac{\partial c_t(x)}{\partial t} = b(x) \star c_t(x) = \int_0^1 b(1-y) \cdot c_t(x)\, dy = c_t(x) \cdot \int_0^1 b(1-y)\, dy = B \cdot c_t(x)$$

Next take

$$\frac{\partial c_t(x)}{\partial t} = \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

as in **Remark 6.2**. Setting the two expressions equal implies

$$B \cdot c_t(x) = \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

or

$$c_t(x) = \mathbf{C} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

This piece of information allows us to derive the specific form of the function $\mathbf{T}(\mathrm{t})$, because

(1) $c_t(x) = p_\infty(x) + \mathbf{T}(\mathrm{t}) \cdot \mathbf{X}(\mathrm{x}) \cdot \mathbf{A}$ from **Remark 6.1**.
(2) $c_t(x) = \mathbf{C} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$ from above.

We have the partial, separable differential equation:

$$p_\infty(x) + \mathbf{T}(\mathrm{t}) \cdot \mathbf{X}(\mathrm{x}) \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

or

$$\left(\mathbf{C} \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} - \mathbf{A} \cdot \mathbf{T}(\mathrm{t}) \right) = \frac{p_\infty(x)}{\mathbf{X}(\mathrm{x})} = k$$

and

$$\mathbf{T}(\mathrm{t}) = \alpha e^{\mathbf{A}t/\mathbf{C}} - \frac{k}{\mathbf{A}}$$

The interesting thing is we know several properties of $\mathbf{T}(\mathrm{t})$ from **Remark 5.8**, in particular, that

(1) $\lim_{t\to\infty} \mathbf{T}(\mathrm{t}) = 0$
(2) $\lim_{t\to\infty} \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} = 0$
(3) $\mathbf{T}(\mathrm{t})$ is never zero

The first condition already shows us an inconsistency, for if we evaluate the limit:

$$\lim_{t\to\infty} \mathbf{T}(\mathrm{t}) = \lim_{t\to\infty} \left(\alpha e^{\mathbf{A}t/\mathbf{C}} - \frac{k}{\mathbf{A}} \right) = \alpha \lim_{t\to\infty} \left(e^{\mathbf{A}t/\mathbf{C}} \right) - \frac{k}{\mathbf{A}} = 0$$

forces the ratio $\frac{\mathbf{A}}{\mathbf{C}}$ to be negative, and thus the limit must be 0. We are therefore left with

$$-\frac{k}{\mathbf{A}} = 0$$

or $k = 0$. Now recall however that

$$\frac{p_\infty(x)}{\mathbf{X}(\mathrm{x})} = k$$

from the partial differential equation, so that the conjunction of the two conditions gives us the contradiction that

$$p_\infty(x) = 0$$

More explicitly the contradiction results from the fact that $p_\infty(x)$ should be a *Pasquali patch* and therefore integrable to 1. We must conclude

$$\frac{\partial c_t(x)}{\partial t} \not\propto c_t(x)$$

Next, take the alternative proposition that

$$\frac{\partial c_t(x)}{\partial t} \propto c_t(x) + q(x)$$

in which we have

$$\frac{\partial c_t(x)}{\partial t} = b(x) \star (c_t(x) + q(x)) = B \cdot c_t(x) + B \cdot q(x)$$

As before, take from **Remark 6.2** the fact that

$$\frac{\partial c_t(x)}{\partial t} = \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

Setting both equations equal we get:

$$B \cdot (c_t(x) + q(x)) = \mathbf{A} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t}$$

and

$$c_t(x) = \mathbf{C} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} - q(x)$$

This expression together with the expression for $c_t(x)$ in **Remark 6.1** gives:

$$p_\infty(x) + \mathbf{T}(\mathrm{t}) \cdot \mathbf{X}(\mathrm{x}) \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{X}(\mathrm{x}) \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} - q(x)$$

Separation yields:

$$\left(\mathbf{C} \cdot \frac{\partial \mathbf{T}(\mathrm{t})}{\partial t} - \mathbf{A} \cdot \mathbf{T}(\mathrm{t})\right) = \frac{p_\infty(x) + q(x)}{\mathbf{X}(\mathrm{x})} = k$$

and again

$$\mathbf{T}(\mathrm{t}) = \alpha e^{\mathbf{A}t/\mathbf{C}} - \frac{k}{\mathbf{A}}$$

As before we argue that $k = 0$ using the first condition of $\mathbf{T}(\mathrm{t})$, but now there is no contradiction in the equation $\frac{p_\infty(x)+q(x)}{\mathbf{X}(\mathrm{x})} = k = 0$ since it is perfectly okay to have

$$p_\infty(x) = -q(x)$$

It is a routine check that the second and third conditions of $\mathbf{T}(\mathrm{t})$ remain true, and now we know for certain that the specific form of $\mathbf{T}(\mathrm{t}) = a \cdot e^{-rt}$ as we had ably surmised before when discussing the *Pasqualian*. Thus we have proven the forward implication:

$$\frac{\partial c_t(x)}{\partial t} \propto c_t(x) + q(x) \Rightarrow \mathbf{T}(\mathrm{t}) = a \cdot e^{-rt}$$

□

7.2. **Propagating the Probability Distribution Wavevector: The *Pasqualian*.** Let us now examine how the probability distribution $c_t^\circ(x)$ changes in time.

Claim 7.2. ***(June 9, 2013)*** *A continuous-in-time* Pasquali patch $p(x, y, t)$ *(a* Pasqualian*), so that*

$$\lim_{t\to\infty} \frac{\partial P^t}{\partial t} = 0$$

and for which

$$\frac{\partial c_t^\circ(x)}{\partial t} = -c_t(x) \star M$$

M is a function of x, y, has form

$$p(x, y, t) = M \cdot e^{-t} + C(x, y)$$

Proof (May 19, 2013) Recall the expression from **Proof 1** of **Claim 4.2**, where we have:

$$c_{t+1}(x) - c_t(x) = c_0(x) \star \left(P^{t+1} - P^t\right)$$

which we now write

$$\Delta c_t(x) = c_0(x) \star \Delta P^t$$

The continuous time analogue is:

$$\frac{\partial c_t^\circ(x)}{\partial t} = c_0(x) \star \frac{\partial P^t}{\partial t}$$

Now let us rewrite $c_t^\circ(x)$ as $\psi(x, t)$ to make explicit the connection to quantum mechanics. Thus we have

$$\frac{\partial \psi(x, t)}{\partial t} = \psi(x, 0) \star \frac{\partial P^t}{\partial t}$$

In BraKet notation the connection is absolutely clear (minus the complex variables), we have:

$$\frac{\partial \langle \psi(x, t)|}{\partial t} = \langle \psi(x, 0)| \frac{\partial P^t}{\partial t}$$

In the Shrödinger time-dependent picture,

$$\frac{\partial \langle \psi(x, t)|}{\partial t} = \langle \psi(x, t)| M$$

that is, the state wavevector (in this case a probability distribution) time derivative is proportional to the state wavevector (also a probability distribution by **Claim 2.13**) itself (typically, the wavefunction M is static and not changing in time, though this may not be the case). Now recall that by **Claim 2.28** (in BraKet notation and continuous time)

$$\langle \psi(x, t)| = \langle \psi(x, 0)| P^t$$

Putting these last three equations together, we get

$$\langle \psi(x, 0)| \frac{\partial P^t}{\partial t} = \langle \psi(x, t)| M = \langle \psi(x, 0)| P^t M$$

which we can simplify as

$$\frac{\partial P^t}{\partial t} = P^t M$$

and solves P^t as an exponential

$$p(x, y, t) = M \cdot e^t + C(x, y)$$

(this is the Shrödinger solution of the wavefunction!). In order for the solution to have stationary states, however, we have to require that the time derivative actually be asymptotic to 0 (see **Corollary 2.24**). Thus we must really have

$$p(x, y, t) = M \cdot e^{-t} + C(x, y)$$

which implies in turn by reverse argument that

$$\frac{\partial \langle \psi(x, t)|}{\partial t} = - \langle \psi(x, t)| M$$

In particular, then, now we are aligned with **Corollary 4.1** for continuous time. We may rename the *Pasquali patch* a *Pasqualian*, in line with the subject-matter nomenclature (Hamiltonian, e.g.). □

Remark 7.1 (Wave-Particle Duality: Resolving the Paradox)**.** ***(May 19, 2013)*** *We can see now that the* Pasqualian *pushes the original wavevector* $\langle \psi(x, 0)|$*, really a probability distribution, forward in time. But the* Pasqualian *wavefunction describes the probability of transition from a (certain) state to another (certain) state at any point in time. In fact, it tells us how the proportions (number of photons) accumulate in space through time. Thus, under this light, the wave interpretation of wave-particle duality arises from a consideration of probability (number of photon) accumulations. The fact is that photons (electrons, or other particles) do not interfere in the ordinary sense of the word (much less singly) with themselves... it is their stacked accumulations that follows the natural probability/frequency law. Photons (electrons, etc.) seem perfectly corpuscular, their wave-like behavior arising from transition-probability dynamics. In this interpretation, waves arise simply from summing position probabilities of particles. Thus we resolve the wave-particle duality by thinking of a photon (electron, etc.) not as possessing both wave and particle properties, but by understanding the interference pattern as aggregate accumulations of many particles as they interact in time following the natural aggregation law. That is, under this interpretation the understanding is that a photon (electron, etc.) is a full particle which, in conjunction with many others, together bundle and spread this way and that along the space of the system according to the laws of probability (the law of step-wise accumulation). This notion we shall call Particle Singularity for purposes of contrast.*

Remark 7.2 (Reinterpretation of the Least Action Principle for the Path of a Particle)**.** ***(May 19, 2013)*** *Using the* Pasqualian*, we can reinterpret the least action principle by stating that a particle will follow the path that* maximizes the position probability *along the totality of (probability) wavevectors of the system (plus minus perturbations).*

8. Measuring Entropy

Remark 8.1. ***(September 5, 2014)*** *Recall from* ***Corollary 2.24*** *that, for a system described by* Pasquali patches*, if there exists a probability distribution so that it is fixed for any of the powers of the generating* Pasquali patch*, this implied that powers of the generator* Pasquali patch *would necessarily converge (and vice versa), a concept we dubbed* entropy *because, conceptually, if no energy were inputed into the system the system would have to settle. We therefore see the need for creating a way to measure the entropy state in which the system is: if we are far away from the limiting surface, there is much progress to be made toward it. If we are* at *the stationary surface, we are at the final possible state.*

8.1. **Entropy of *Pasquali Patches*.**

Definition 8.1. ***(September 5, 2014)*** *Take any* Pasquali patch *in the collection* $p_n(x, y), n \in \mathbb{Z}^+$*. Define the functions* $S_1, S_2 \colon \mathbb{Z}^+ \to [0, \infty)$ *in the following way:*

$$S_1(n) = \int_{[0,1]^2} |p_n(x, y) - p_\infty(x)| \, dA$$

and

$$S_2(n) = \sqrt{\int_{[0,1]^2} (p_n(x, y) - p_\infty(x))^2 \, dA}$$

$S_1(n)$ *measures the absolute deviations of* Pasquali patches *from the stationary surface, where* $S_2(n)$ *emphasizes larger deviations over smaller deviations and standardizes them (hence* standard entropy*). It seems clear that if* $S_1(n) = S_2(n) = 0$*, the system is at its stationary state, and the entropy is the largest it can be.*

8.2. **Entropy of the *Pasqualian*.**

Definition 8.2. ***(September 5, 2014)*** *Take a* Pasqualian. *Define the functions* $S_1, S_2 \colon \mathbb{R} \to [0, \infty)$ *so that:*

$$S_1(t) = \int_{[0,1]^2} |p(x, y, t) - p_\infty(x)| \; dA$$

and

$$S_2(t) = \sqrt{\int_{[0,1]^2} (p(x, y, t) - p_\infty(x))^2 \; dA}$$

Claim 8.1. ***(February 7, 2015)*** *// IN PROGRESS*

Proof Recall the expression of the *Pasqualian* from **Remark 5.8**:

$$p(x, y, t) = \mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x)$$

It seems clear that, plugging into **Definition 8.2**,

$$\begin{aligned} S_1(t) &= \int_{[0,1]^2} |p(x, y, t) - p_\infty(x)| \; dA \\ &= \int_{[0,1]^2} |\mathbf{M}(\mathrm{x}, \mathrm{y}) \cdot \mathbf{T}(\mathrm{t}) + p_\infty(x) - p_\infty(x)| \; dA \\ &= |\mathbf{T}(\mathrm{t})| \cdot \int_{[0,1]^2} |\mathbf{M}(\mathrm{x}, \mathrm{y})| \; dA \\ &= |\mathbf{T}(\mathrm{t})| \cdot \left(\int_0^1 |\mathbf{X}(\mathrm{x})| \; dx \right) \cdot \left(\int_0^1 |\mathbf{Y}(\mathrm{y})| \; dy \right) \end{aligned}$$

and, analogously,

$$\begin{aligned} S_2(t) &= \sqrt{(\mathbf{T}(\mathrm{t}))^2 \cdot \int_{[0,1]^2} (\mathbf{M}(\mathrm{x}, \mathrm{y}))^2 \; dA} \\ &= \mathbf{T}(\mathrm{t}) \cdot \left(\sqrt{\int_0^1 (\mathbf{X}(\mathrm{x}))^2 \; dx} \right) \cdot \left(\sqrt{\int_0^1 (\mathbf{Y}(\mathrm{y}))^2 \; dy} \right) \end{aligned}$$

with $\mathbf{T}(\mathrm{t})$ is strictly positive (from the squaring and taking the positive square root).

Recall from **Remark 5.6**, that the equation of probability propagation using **Construction 2.1** is

$$p(x, y, t) = \overbrace{\left[\left(f_1(x) - \frac{F_1}{F_2} f_2(x) \right) \cdot (g_1(y) - B) \cdot \frac{1}{T(1)} \right] \cdot T(t)}^{M(x,y) \cdot T(t)} + \overbrace{\frac{f_2(x)}{F_2} - \left(\frac{f_2(x) F_1}{F_2} - f_1(x) \right) B}^{p_\infty(x)}$$

Using $S_2(t)$, the relevant expression becomes:

$$\begin{aligned} S_2(t) &= \\ &= \int_{[0,1]^2} (M(x, y) \cdot T(t))^2 \; dA \\ &= \int_{[0,1]^2} \left(\left[\left(f_1(x) - \tfrac{F_1}{F_2} f_2(x) \right) \cdot (g_1(y) - B) \cdot \tfrac{1}{T(1)} \right] \cdot T(t) \right)^2 \; dA \end{aligned}$$

□

9. Applications

Example 9.1. ***(May 5, 2013)*** *Suppose we have an idealized canal of width* 1*, on which a fluid flow has been established in some remote past. Let us focus solely on the dynamics of the surface. Pick a spot along the canal which we will call* t_0*. Next pick a spot* $y_0 \in [0, 1]$ *along the width of the canal, which we will monitor. Pick a second spot* t_1 *down the canal, some distance from the original spot we picked. Now let us assume that, up the canal at some remote point, a paper boat has been released. We will only care about the boat if the boat passes through* (t_0, y_0) *which we have picked, and we will write down the resultant position at* t_1*. Let us do this a number of times with any number of boats, and obtain a distribution of the position of the boat at* t_1*, saving it. Next let us repeat the experiment, this time focusing on* y_0'*, save the resultant distribution at* t_1*, and so on and so forth, until we are comfortable having mapped the totality of positions at* t_0*. Let us next put together (stack, respecting the natural order of* y*) all the distributions we obtained at* t_1*. We now have a discrete surface which we can smooth to obtain a* Pasquali patch.

Let us now look at position t_2 *which is the same distance as* t_1 *is from* t_0*. Having defined the dynamics of the system (from a single* Pasquali patch*), the dynamics at* t_2 *can be theoretically described by* P^2*. We can therefore ascertain the probability that we will find the boat at* t_2 *along the width of the canal. In fact, at* t_n*,* n *very large, we can ascertain the probability that the boat will be at any position along the width. It should be close to* P^∞*. More importantly, a great distance from the origin (any distance, not necessarily a distance* $n \cdot \Delta t_n$*), the position probability is aptly described by* P^∞*. See **Figure 7** and **Figure 8**.*

This simple thought experiment brings about several questions. What if the dynamics of the surface system are described by the Pasquali patch*, but at points which are not a distance* Δt_n *apart? In other words, what if the description is apt but at points that are not linear in distance? This curious situation suggests a time anomaly, and therefore a manner in which we can* measure *time warps (by measuring the actual time differences between* Pasquali patches*). See **Figure 9**.*

Next, we looked at the surface dynamics of the system. If we add a depth variable to the canal, we can in theory produce a Pasquali cube*, which would measure the dynamics of any point on the* $[0,1]\times[0,1]$ *cross-section a discrete distance down the canal (and any distance very far from our origin).*

*A third question arises when we consider the same canal, but whose width opens by a scalar (linear) amount a distance from our chosen origin. There is no reason we cannot "renormalize" the width (set it equal to 1 again) at a point some set distance from our chosen origin, and proceed with our analysis as before. See **Figure 10**.*

A fourth question arises when we begin to think that the canal width does not open (or close) linearly.

10. Remarks that Challenge Established Notions

Remark 10.1. ***(January 13, 2014)*** *The statistical description of a dynamical system (as one described by a generator* Pasquali patch*) really does give us a lot of power in computing the probable* position *of a particle (photon, electron) moving in space at different (integer) time intervals. If the quantum mechanical supposition of time having a minimum discreteness (Planck-time) is correct, we can find the* finest Pasquali patch generator *that will give a complete description of the dynamical system. Any* Pasquali patch *generator descriptive of the system which is not this* first *will generate an accurate, yet less refined (*coarser*) version of the system (this is what we mean by **Claim 1.13**, in that such* Pasquali patch *will be* contained *in the finest description, yet is not the finest), and in fact either system of course converges to the same steady state (this is what is meant by **Claim 2.34**). If we are able to find a continuous description (like the Shrödinger equation, via a* Pasqualian*) of such system then we are in luck (this description would be the finest, though non-discrete, description), and I speculate though I cannot be sure yet that either discrete descriptions will be contained in such.*

Whatever the description of the dynamical system via a generator Pasquali patch *(or a* Pasqualian*), each* Pasquali patch *represents the transition (position) probabilities of a particle (photon, electron) moving within that system. If we suppose that the particle moves with same velocity (take photons in vacuum as an example), then each* Pasquali patch *power is descriptive of the transition-position probability at equally spaced spacial or distance intervals. Though I've remarked about this before, if it were the case that, for a particle with a particular (steady) velocity, the* Pasquali patch *power is not exactly equally spaced in distance intervals, it must mean that the arrow of time is bent (time is moving faster for smaller-spaced intervals, slower for longer-spaced). We have not yet described accelerating particles but at present that is not of our interest.*

However, we can tell if time is passing so long as each Pasquali patch *description is different at each (equal or unequal) interval. If a single* Pasquali patch *were to describe the system at EACH distance interval, there is no way to know if time is moving at all. Take for example **Claim 2.35**. We had the collection* $\mathbb{Q} = \{Q^1, Q^2, \dots, Q^k, \dots\}_{k\in\mathbb{Z}^+}$ *with* $Q = q(x, y)$ *being an explicit function of y and converging to* $Q^\infty = 1$*. We could track the time-distance interval via the* Pasquali patch *power, so that* 1 *was the first distance interval from start (we take it as given that time is not being bent, so that a fixed distance implies the passage of 1 unit of time),* 2 *was the second distance interval (2 units of time), and so on, and we could tell if time were bent if each power were descriptive of different distance intervals. Furthermore, since each* $Q \in \mathbb{Q}$ *is different, this implies each position transition probability is different and the system is in movement.*

This is definitely not the case with the collection $\mathbb{P} = \{P^1 = 1, P^2 = 1, \dots, P^j, \dots\}_{j\in\mathbb{Z}^+}$ *which also converges to* $P^\infty = 1$*. Since at each distance interval the movement probability is the same (uniform), one cannot be convinced that each power represents a distance interval equal to equally spaced time intervals or different-spaced time intervals. Where we could with* $\mathbb{Q}$ *ascertain that time was moving, we cannot with* $\mathbb{P}$*. The statistical description cannot tell if the system is frozen.*

*When a system has reached the steady state (which, is the highest entropy state! See **Corollary 2.24**), there is no way to tell if time flows, as the statistical description is and forever will be unchanging. Recall that the steady state for a* Pasquali patch *is always a function of x alone, say* $p(x)$*. Furthermore recall that any power of* $p(x)$ *is always* $p(x)$ *itself (see **Claim 2.11**). We reach an impasse: is time flowing normally, faster than what is conventional, slower? At such a point it is impossible to say, at least from the statistical point of view. We would have to track particles individually in order to ascertain if they have deviated their path at all (it could be the case that they shifted to all positions with equal probability or in the shape of* $p(x)$*, e.g., but we cannot be sure of either situation).*

11. Relevant Generalizations

11.1. **Equivalencies of the Star Operator.**

Claim 11.1. ***(December 29, 2014)*** *The star operation*

$$f(x,y) \star g(x,y) = j\left(f(1-y,z) \diamond g(x,y)\right) = j\left(\int_{y=0}^{y=1} f(1-y,z)\cdot g(x,y)\,dy\right)$$

can also be written

$$f(x,y) \star g(x,y) = j\left(\int_{y=0}^{y=1} f(y,z)\cdot g(x,1-y)\,dy\right)$$

In particular,

$$f(x) \star g(x,y) = \int_0^1 f(y)\cdot g(x,1-y)\,dy$$

Proof Define $s = 1-y$. Thus we can write

$$\begin{aligned} f(x,y) \star g(x,y) &= j\left(\textstyle\int_{1-s=0}^{1-s=1} f(s,z)\cdot g(x,1-s)\,d\,(1-s)\right) \\ &= j\left(-\textstyle\int_{s=1}^{s=0} f(s,z)\cdot g(x,1-s)\,ds\right) \\ &= j\left(\textstyle\int_{s=0}^{s=1} f(s,z)\cdot g(x,1-s)\,ds\right) \end{aligned}$$

Since the s is a dummy variable, we can use y instead, and

$$f(x,y) \star g(x,y) = j\left(\int_0^1 f(y,z)\cdot g(x,1-y)\,dy\right)$$

Thus, where we have $f(x) \star g(x,y)$, we have the alternative expression

$$f(x) \star g(x,y) = \int_0^1 f(y)\cdot g(x,1-y)\,dy$$

□

11.2. **Surface Trace or *str*.**

Definition 11.1. ***(March 3, 2013)*** *Take the function $h\colon [0,1]^2 \to \mathbb{R}$. Let the* surface trace *be the value of the integral of the diagonal $y = -x+1$ or $x = 1-y$ of such surface. In other words, it is:*

$$\operatorname{str}\left[h(x,y)\right] = \int_0^1 h(x,1-x)\,dx = \int_0^1 h(1-y,y)\,dy$$

11.3. **The Eigenvalue Question.**

Consider the general eigenvalue question

$$j(x) \star h(x,y) = \lambda\, j(x)$$

for general (well-behaved) functions $j\colon [0,1] \to \mathbb{R}$ and $h\colon [0,1]^2 \to \mathbb{R}$.

Claim 11.2. ***(July 24, 2013)*** *If λ is an eigenvalue of $h(x,y)$, then λ^n is an eigenvalue of $h_n(x,y)$, the nth power of $h(x,y)$, $n \in \mathbb{Z}^+$.*

Proof by Induction Let

$$j(x) \star h(x,y) = \lambda\, j(x)$$

Suppose that

$$j(x) \star h_k(x,y) = \lambda^k\, j(x)$$

Then

$$\begin{aligned} j(x) \star h_{k+1}(x,y) &= \left(j(x) \star h_k(x,y)\right) \star h(x,y) \\ &= \left(\lambda^k\, j(x)\right) \star h(x,y) \\ &= \lambda^k\left(j(x) \star h(x,y)\right) \\ &= \lambda^k \cdot \lambda \cdot j(x) \\ &= \lambda^{k+1}\, j(x) \end{aligned}$$

We have made liberal use of **Claim 1.8 Associativity of the Star Product**. □

Corollary 11.3. ***(July 24, 2013)*** *This is a restatement of* ***Corollary 2.16*** *for* Pasquali patches. *Provided we can find $a(x)$ so that it is fixed under the* Pasquali patch $p(x, y)$*, that is,*

$$a(x) \star p(x, y) = a(x)$$

and $\lambda = 1$, it follows that $\lambda = 1$ for all $p_n(x, y)$, $n \in \mathbb{Z}^+$.

Proof Simply apply **Claim 11.2** to the fact that *Pasquali patches* have $\lambda = 1$ eigenvalue (provided $a(x)$ exists). □

11.4. Specific Finite-Dot-Product Surfaces.

Definition 11.2. ***(March 3, 2013)*** *A specific finite-dot-product surface is a function $h\colon [0, 1]^2 \to \mathbb{R}$ such that:*

$$h(x, y) = \sum_{k=1}^{n} f_k(x) g_k(y) = \mathbf{f}(\mathrm{x}) \cdot \mathbf{g}(\mathrm{y})$$

11.4.1. *Specific Finite-Dot-Product Surface Trace (*str*).*

Claim 11.4. ***(March 3, 2013)*** *The specific finite-dot-product surface trace is:*

$$\mathrm{str}\left[h(x, y)\right] = \sum_{k=1}^{n} C_{k,k} = \mathrm{tr}\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \end{bmatrix}$$

where $C_{k,k} = f_k(x) \star g_k(x)$.

Proof by Induction We have:

$$\mathrm{str}\left[h(x, y)\right] = \int_0^1 h(x, 1 - x)\, dx = \int_0^1 h(1 - y, y)\, dy$$

by **Definition 11.1**. We can think of the specific finite-dot-product surface

$$h(x, y) = f_1(x) g_1(y)$$

and readily calculate the trace str in this way:

$$\begin{aligned} \mathrm{str}\left[h(x, y)\right] = \int_0^1 h(1 - y, y)\, dy &= \int_0^1 f_1(1 - y) g_1(y)\, dy \\ &= \int_0^1 f_1(1 - y) g_1(y)\, dy \\ &= f_1(x) \star g_1(y) \\ &= C_{1,1} \\ &= \mathrm{tr}\left[C_{1,1}\right] \end{aligned}$$

This constitutes the base case. So assume that the mth case is true, and

$$\begin{aligned} \mathrm{str}\left[h(x, y)\right] = \sum_{k=1}^{m} C_{k,k} &= \mathrm{tr}\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,m} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m,1} & C_{m,2} & \cdots & C_{m,m} \end{bmatrix} \\ &= \sum_{k=1}^{m} \left(\int_0^1 f_k(1 - y) g_k(y)\, dy \right) \\ &= \int_0^1 \left(\sum_{k=1}^{m} f_k(1 - y) g_k(y) \right) dy \end{aligned}$$

We now show that the $m + 1$ case works as well. So let

$$h(x, y) = \sum_{k=1}^{m+1} f_k(x) g_k(y) = \sum_{k=1}^{m} f_k(x) g_k(y) + f_{m+1}(x) g_{m+1}(y)$$

Then the surface trace can be calculated as

$$\begin{aligned}
\operatorname{str}[h(x,y)] = \int_0^1 h(1-y,y)\,dy &= \int_0^1 \left(\sum_{k=1}^{m} f_k(1-y)g_k(y) + f_{m+1}(1-y)g_{m+1}(y)\right) dy \\
&= \int_0^1 \left(\sum_{k=1}^{m} f_k(1-y)g_k(y)\right) dy + \int_0^1 f_{m+1}(1-y)g_{m+1}(y)\,dy \\
&= \sum_{k=1}^{m} C_{k,k} + C_{m+1,m+1} \\
&= \sum_{k=1}^{m+1} C_{k,k} \\
&= \operatorname{tr}\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,m} & C_{1,m+1} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,m} & C_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{m,1} & C_{m,2} & \cdots & C_{m,m} & C_{m,m+1} \\ C_{m+1,1} & C_{m+1,2} & \cdots & C_{m+1,m} & C_{m+1,m+1} \end{bmatrix}
\end{aligned}$$

□

11.4.2. *Specific Finite-Dot-Product Surface Eigenvalues.*

Claim 11.5. ***(March 25, 2011)*** *Let* $e\colon [0,1] \to \mathbb{R}$ *be smooth and well-behaved, and* $h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y)$ *with* $h\colon [0,1]^2 \to \mathbb{R}$ *likewise. Then there are* ***two*** λ *values that satisfy*

$$e(x) \star h(x,y) = \lambda\, e(x)$$

provided

$$C_{1,1} = f_1(x) \star g_1(y)$$

$$C_{1,2} = f_1(x) \star g_2(y)$$

$$C_{2,1} = f_2(x) \star g_1(y)$$

$$C_{2,2} = f_2(x) \star g_2(y)$$

converge.

Proof In **Claim 11.5**, the equation

$$e(x) \star h(x,y) = \lambda\, e(x)$$

can be written out specifically as

$$\int_0^1 e(1-y)h(x,y)\,dy = \lambda\, e(x)$$

More explicitly, this is:

$$\begin{aligned}
\lambda\, e(x) &= \int_0^1 e(1-y)\left(f_1(x)g_1(y) + f_2(x)g_2(y)\right) dy \\
&= f_1(x)\int_0^1 e(1-y)g_1(y)\,dy + f_2(x)\int_0^1 e(1-y)g_2(y)\,dy \\
&= B_1 f_1(x) + B_2 f_2(x)
\end{aligned}$$

where B_1, B_2 are constants. If we divide by λ as

$$e(x) = \frac{B_1}{\lambda} f_1(x) + \frac{B_2}{\lambda} f_2(x)$$

then the equation must hold provided $\lambda \neq 0$. So we have excluded an eigenvalue right from the start.

We can systematically write the derivatives of $e(x)$:

$$\begin{array}{rcl} e(x) & = & \frac{B_1}{\lambda}f_1(x) + \frac{B_2}{\lambda}f_2(x) \\ e'(x) & = & \frac{B_1}{\lambda}f_1'(x) + \frac{B_2}{\lambda}f_2'(x) \\ e''(x) & = & \frac{B_1}{\lambda}f_1''(x) + \frac{B_2}{\lambda}f_2''(x) \\ \vdots & \vdots & \vdots \\ e^k(x) & = & \frac{B_1}{\lambda}f_1^k(x) + \frac{B_2}{\lambda}f_2^k(x) \\ \vdots & \vdots & \vdots \end{array}$$

again with $\lambda \neq 0$. We want to calculate the constants B_1, B_2, to see if they are restricted in some way by a formula, and we do this by integrating by parts as we did before. Thus, we have that if

$$B_1 = \int_0^1 e(1-y)g_1(y)\,dy$$

the tabular method gives:

Derivatives	Integrals
$e(1-y)$	$g_1(y)$
$-e'(1-y)$	$G_1^1(y)$
$e''(1-y)$	$G_1^2(y)$
$\vdots$	$\vdots$

and so,

$$\begin{aligned} B_1 &= \int_0^1 e(1-y)g_1(y)\,dy \\ &= e(1-y)G_1^1(y)\big|_0^1 + e'(1-y)G_1^2(y)\big|_0^1 + \dots \\ &= \sum_{i=0}^{\infty} e^i(1-y)G_1^{i+1}(y)\Bigg|_0^1 \end{aligned}$$

if we remember the alternating sign of the multiplications, and we are allowed some leeway in notation. Ultimately, this last bit means: $\sum_{i=0}^{\infty} e^i(0)G_1^{i+1}(1) - \sum_{i=0}^{\infty} e^i(1)G_1^{i+1}(0)$. Since we have already explicitly written the derivatives of $e(x)$, the $e^i(0), e^i(1)$ derivatives can be written as

$$\frac{B_1}{\lambda}f_1^i(0) + \frac{B_2}{\lambda}f_2^i(0)$$

and

$$\frac{B_1}{\lambda}f_1^i(1) + \frac{B_2}{\lambda}f_2^i(1)$$

respectively. We have then:

$$B_1 = \sum_{i=0}^{\infty}\left(\frac{B_1}{\lambda}f_1^i(0) + \frac{B_2}{\lambda}f_2^i(0)\right)G_1^{i+1}(1) - \sum_{i=0}^{\infty}\left(\frac{B_1}{\lambda}f_1^i(1) + \frac{B_2}{\lambda}f_2^i(1)\right)G_1^{i+1}(0)$$

Since we aim to solve for B_1, multiplying by λ makes things easier, and also we must rearrange all elements with B_1 in them, so we get:

$$\lambda B_1 = B_1\sum_{i=0}^{\infty}\left(f_1^i(0)G_1^{i+1}(1) - f_1^i(1)G_1^{i+1}(0)\right) + B_2\sum_{i=0}^{\infty}\left(f_2^i(0)G_1^{i+1}(1) - f_2^i(1)G_1^{i+1}(0)\right)$$

Subtracting both sides the common term and factoring the constant we endeavor to solve for, we get:

$$\left(\lambda - \sum_{i=0}^{\infty}\left(f_1^i(0)G_1^{i+1}(1) - f_1(1)G_1^{i+1}(0)\right)\right)B_1 = B_2\sum_{i=0}^{\infty}\left(f_2^i(0)G_1^{i+1}(1) - f_2^i(1)G_1^{i+1}(0)\right)$$

or

$$B_1 = \frac{B_2\sum_{i=0}^{\infty} f_2^i(1-y)G_1^{i+1}(y)\Big|_0^1}{\lambda - \sum_{i=0}^{\infty} f_1^i(1-y)G_1^{i+1}(y)\Big|_0^1} = \frac{B_2\left(f_2(x) \star g_1(y)\right)}{\lambda - \left(f_1(x) \star g_1(y)\right)} = \frac{B_2 C_{2,1}}{\lambda - C_{1,1}}$$

A similar argument for B_2 suggests

$$B_2 = \frac{B_1 \sum_{i=0}^{\infty} f_1^i(1-y)G_2^{i+1}(y)\Big|_0^1}{\lambda - \sum_{i=0}^{\infty} f_2^i(1-y)G_2^{i+1}(y)\Big|_0^1} = \frac{B_1\left(f_1(x)\star g_2(y)\right)}{\lambda - (f_2(x)\star g_2(y))} = \frac{B_1 C_{1,2}}{\lambda - C_{2,2}}$$

where the new constants introduced emphasizes the expectation that the sums (or integrals) converge. Plugging in the one into the other we get:

$$B_1 = \frac{\left(\frac{B_1 C_{1,2}}{\lambda - C_{2,2}}\right) C_{2,1}}{\lambda - C_{1,1}} = \frac{B_1 C_{1,2} C_{2,1}}{(\lambda - C_{2,2})(\lambda - C_{1,1})}$$

and now we have additional restrictions on lambda: $\lambda \neq C_{2,2}$ and $\lambda \neq C_{1,1}$. Furthermore, the constant B_1 drops out of the equation, suggesting these constants can be anything we can imagine (all of $\mathbb{R}$ without restriction), but then we have the constraint:

$$(\lambda - C_{2,2})(\lambda - C_{1,1}) = C_{1,2}C_{2,1}$$

(Notice how this equation can be put in determinant form!

$$\det \begin{vmatrix} C_{1,1} - \lambda & C_{1,2} \\ C_{2,1} & C_{2,2} - \lambda \end{vmatrix} = 0$$

This form of the equation becomes the basis of **Claim 11.6**.)

Expanding the equation suggests:

$$\lambda^2 - (C_{2,2} + C_{1,1})\lambda + (C_{1,1}C_{2,2} - C_{1,2}C_{2,1}) = 0$$

which we can solve by the quadratic equation of course, as:

$$\lambda_{1,2} = \frac{(C_{2,2} + C_{1,1}) \pm \sqrt{(C_{2,2} - C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}$$

So not only is λ not equal to many values, it is incredibly restricted to two of them.

Now the constants $C_{1,1}, C_{1,2}, C_{2,1}, C_{2,2}$ have been expressed in terms of the integration-by-parts sums in the expectation that subsequent derivatives of $f_{1,2}(x)$ will eventually vanish (or are periodically 0 [3]). There is nothing to stop us from redefining them in terms of $g_{1,2}(y)$ derivatives instead, if these were to vanish quicker or were to force the sum convergence where the derivatives of $f_{1,2}(x)$ did not.[4]

Now, figuring out the eigenfunctions $e(x)$ that go together with these eigenvalues is an exercise in finding constants B_1 and B_2 for a given eigenvalue. [5] □

Claim 11.6. ***(January 26, 2013)*** *The allowable values of λ is equal to the number of (pair) function terms*

$$h(x,y) = \sum_{k=1}^{n} f_k(x)g_k(y)$$

has, provided pairwise star products between functions of x and functions of y converge. In particular, these can be found by

$$\det|A - \lambda I| = 0$$

where A is an $n \times n$ matrix of pairwise star products $f_i(x) \star g_j(y)$ (we call such constants $C_{i,j}$) with $i, j \in \{1 \dots n\}$, and I is the identity matrix. This creates a characteristic equation, in direct analogy to how we obtain eigenvalues in linear algebra contexts.

---

[3] Need to clarify

[4] For this reason it may be convenient to leave the integration-by-parts method open-ended by rewriting and redefining shorthand

$$f_1(x) \star g_1(y) = C_{1,1}$$
$$f_2(x) \star g_1(y) = C_{2,1}$$
$$f_1(x) \star g_2(y) = C_{1,2}$$
$$f_2(x) \star g_2(y) = C_{2,2}$$

like we did.

[5] Need to show

Proof The base case has already been shown in **Claim 11.5**. Suppose that

$$h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y) + \ldots + f_k(x)g_k(y)$$

and as before, we are looking for

$$e(x) \star h(x,y) = \lambda\, e(x)$$

Arguing similarly as in **Claim 11.5**, we end up with the linear system:

$$\begin{aligned} \lambda B_1 &= B_1C_{1,1} + B_2C_{2,1} + \ldots + B_kC_{k,1} \\ \lambda B_2 &= B_1C_{1,2} + B_2C_{2,2} + \ldots + B_kC_{k,2} \\ &\vdots \\ \lambda B_k &= B_1C_{1,k} + B_2C_{2,k} + \ldots + B_kC_{k,k} \end{aligned}$$

for which we can write the augmented matrix

$$\left[\begin{array}{cccc|c} C_{1,1}-\lambda & C_{2,1} & \ldots & C_{k,1} & 0 \\ C_{1,2} & C_{2,2}-\lambda & \ldots & C_{k,2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,k} & C_{2,k} & \ldots & C_{k,k}-\lambda & 0 \end{array}\right]$$

Now the determinant of the square matrix must be equal to 0, otherwise there is exactly one solution for constants $B_1, B_2, \ldots, B_k$ (we are specifically looking that these constants be *any* value, so the matrix must be singular and consequently the determinant equal to zero). Thus we have that

$$\det \begin{vmatrix} C_{1,1}-\lambda & C_{2,1} & \ldots & C_{k,1} \\ C_{1,2} & C_{2,2}-\lambda & \ldots & C_{k,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,k} & C_{2,k} & \ldots & C_{k,k}-\lambda \end{vmatrix} = 0$$

The $k+1$th case can be argued similarly. With

$$h(x,y) = f_1(x)g_1(y) + f_2(x)g_2(y) + \ldots + f_k(x)g_k(y) + f_{k+1}(x)g_{k+1}(y)$$

we get:

$$\begin{aligned} \lambda B_1 &= B_1C_{1,1} + B_2C_{2,1} + \ldots + B_kC_{k,1} \\ \lambda B_2 &= B_1C_{1,2} + B_2C_{2,2} + \ldots + B_kC_{k,2} \\ &\vdots \\ \lambda B_k &= B_1C_{1,k} + B_2C_{2,k} + \ldots + B_kC_{k,k} \\ \lambda B_{k+1} &= B_1C_{1,k+1} + B_2C_{2,k+1} + \ldots + B_{k+1}C_{k+1,k+1} \end{aligned}$$

and so we endeavor to solve the augmented matrix:

$$\left[\begin{array}{ccccc|c} C_{1,1}-\lambda & C_{2,1} & \ldots & C_{k,1} & C_{k+1,1} & 0 \\ C_{1,2} & C_{2,2}-\lambda & \ldots & C_{k,2} & C_{k+1,2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C_{1,k} & C_{2,k} & \ldots & C_{k,k}-\lambda & C_{k+1,k} & 0 \\ C_{1,k+1} & C_{2,k+1} & \ldots & C_{k,k+1} & C_{k+1,k+1}-\lambda & 0 \end{array}\right]$$

so that the vector $B_1, B_2, \ldots, B_k, B_{k+1}$ admits an infinity of solutions. We are thus again looking for the singular matrix with determinant zero:

$$\det \begin{vmatrix} C_{1,1}-\lambda & C_{2,1} & \ldots & C_{k,1} & C_{k+1,1} \\ C_{1,2} & C_{2,2}-\lambda & \ldots & C_{k,2} & C_{k+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,k} & C_{2,k} & \ldots & C_{k,k}-\lambda & C_{k+1,k} \\ C_{1,k+1} & C_{2,k+1} & \ldots & C_{k,k+1} & C_{k+1,k+1}-\lambda \end{vmatrix} = 0$$

which we can now simplify to

$$\det|A - \lambda I| = 0$$

as we wanted. □

11.4.3. *Functions on Eigenvalues of Specific Finite-Dot-Product Surfaces.*
sdet.

Definition 11.3 (Specific Finite-Dot-Product Surface Determinant or *sdet*)**.** ***(February 10, 2013)*** *Specifically in a linear-algebra context, the determinant of a square matrix is the product of its eigenvalues. When we talk about the specific finite-dot-product surface*

$$h(x, y) = \sum_{k=1}^{n} f_k(x) g_k(y)$$

*we shall **define** the* surface determinant *or* sdet *as the product of the eigenvalues it generates:*

$$\text{sdet}\,|h(x, y)| = \prod_{k=1}^{n} \lambda_k$$

Claim 11.7. ***(February 10, 2013)*** *The* sdet *for*

$$h(x, y) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

is

$$\text{sdet}\,|h(x, y)| = C_{1,1}C_{2,2} - C_{1,2}C_{2,1} = \det \begin{vmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{vmatrix}$$

Proof This follows directly from the eigenvalue formulation:

$$\lambda_{1,2} = \frac{(C_{2,2} + C_{1,1}) \pm \sqrt{(C_{2,2} - C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}$$

The product is:

$$\begin{aligned} \lambda_1 \cdot \lambda_2 &= \left(\frac{(C_{2,2} + C_{1,1}) + \sqrt{(C_{2,2} - C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2} \right) \cdot \left(\frac{(C_{2,2} + C_{1,1}) - \sqrt{(C_{2,2} - C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2} \right) \\ &= \frac{(C_{2,2} + C_{1,1})^2 - (C_{2,2} - C_{1,1})^2 - 4C_{1,2}C_{2,1}}{4} \\ &= \frac{4C_{1,1}C_{2,2} - 4C_{1,2}C_{2,1}}{4} \\ &= C_{1,1}C_{2,2} - C_{1,2}C_{2,1} \\ &= \det \begin{vmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{vmatrix} \end{aligned}$$

□

Conjecture 11.1. ***(February 23, 2013)*** *The* sdet *of a specific finite surface with n pair functions of x and y terms is:*

$$\text{sdet}\,|h(x, y)| = \det \begin{vmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \end{vmatrix}$$

str (Revisited).

Claim 11.8. ***(March 3, 2013)*** *In the context of specific finite-dot-product surfaces,*

$$\text{str}\,[h(x, y)] = \sum_{k=1}^{n} \lambda_k$$

Proof by Induction From the eigenvalue formulation of **Claim 11.5**, we have

$$\lambda_{1,2} = \frac{(C_{2,2} + C_{1,1}) \pm \sqrt{(C_{2,2} - C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}$$

The sum is:

$$\lambda_1+\lambda_2 = \left(\frac{(C_{2,2}+C_{1,1})+\sqrt{(C_{2,2}-C_{1,1})^2+4C_{1,2}C_{2,1}}}{2}\right)+\left(\frac{(C_{2,2}+C_{1,1})-\sqrt{(C_{2,2}-C_{1,1})^2+4C_{1,2}C_{2,1}}}{2}\right)$$
$$= C_{1,1}+C_{2,2}$$

which is exactly $\mathrm{str}\left[f_1(x)g_1(y)+f_2(x)g_2(y)\right]$. This constitutes the base case. Next assume that it is true that

$$\mathrm{str}\left[\sum_{k=1}^{m} f_k(x)g_k(y)\right] = \sum_{k=1}^{m} C_{k,k} = \sum_{k=1}^{m}\lambda_k$$

Then

$$\mathrm{str}\left[\sum_{k=1}^{m+1} f_k(x)g_k(x)\right] = \sum_{k=1}^{m-1} C_{k,k} + C_{m,m} + C_{m+1,m+1}$$
$$=$$

□

*************************end in-progress*******************************

11.4.4. *Specific Finite-Dot-Product Surface Specific Infinite-Dot-Product Representations.*

Remark 11.1. ***(February 23, 2013)*** *There is no reason why we should restrict ourselves to the study of specific finite-dot-product surfaces, where we could make the leap to specific infinite-dot-product surfaces of the form*

$$h(x,y) = \sum_{k=0}^{\infty} f_k(x)g_k(y)$$

Conjecture 11.2. ***(February 23, 2013) Conjecture* ??** *suggests that the* str *of a specific infinite-dot-product surface is:*

$$\mathrm{str}\left[h(x,y)\right] = \sum_{k=1}^{\infty}\lambda_k = \sum_{k=1}^{\infty} C_{k,k}$$

provided all $C_{k,k}$ converge in their calculation and the sum itself converges.

Example 11.1. ***(February 23, 2013)*** *Recall the Maclaurin series expansion*

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

We can think of this equation as

$$\overbrace{e^x}^{f_1^\circ(x)}\cdot\overbrace{1}^{g_1^\circ(y)} = \overbrace{1}^{f_1(x)}\cdot\overbrace{1}^{g_1(y)} + \overbrace{x}^{f_2(x)}\cdot\overbrace{1}^{g_2(y)} + \overbrace{\frac{x^2}{2!}}^{f_3(x)}\cdot\overbrace{1}^{g_3(y)} + \overbrace{\frac{x^3}{3!}}^{f_4(x)}\cdot\overbrace{1}^{g_4(y)} + \cdots$$

In other words, a specific finite-dot-product surface on the LHS and a specific infinite-dot-product surface on the RHS. The LHS has sum of eigenvalues (eigenvalue)

$$e^x\star 1 = \int_0^1 e^{(1-y)}\,dy = e-1$$

by the trivial case of ***Claim 11.6****, and the RHS is calculated directly by*

$$\begin{aligned} &= \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(1-y)^k}{k!} \right) dy \\ &= \sum_{k=0}^{\infty} \left(\int_0^1 \frac{(1-y)^k}{k!} \, dy \right) \\ &= \sum_{k=0}^{\infty} \left(- \frac{(1-y)^{(k+1)}}{(k+1)k!} \bigg|_0^1 \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \end{aligned}$$

where we have pulled the integral inside the sum due to absolute convergence and the infinite sum of factorial reciprocals therefore of course also converges.[6] *This gives credence to* ***Conjecture 11.2****, since the second row is exactly the* str *of a specific infinite-dot-product surface*

$$\operatorname{str} \left| \sum_{k=0}^{\infty} \frac{x^k}{k!} \right| = 1 \star 1 + x \star 1 + \frac{x^2}{2!} \star 1 + \frac{x^3}{3!} \star 1 + \cdots = \sum_{k=1}^{\infty} \lambda_k$$

Claim 11.9. ***(February 23, 2013)*** *Take the Taylor-expandable function $f(x)$, $f\colon [0,1] \times [0,1] \to \mathbb{R}$*

$$f(x) = \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} (x-a)^i$$

with $a \in [0,1]$. Then its eigenvalue is the convergent series sum

$$\lambda = \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right)$$

with $a \in [0,1]$.

Proof We have that

$$\overbrace{f(x)}^{f_1^\circ(x)} \cdot \overbrace{1}^{g_1^\circ(y)} = \overbrace{\left(\sum_{i=0}^{\infty} \frac{f^i(a)}{i!} (x-a)^i \right)}^{f_1^\circ(x)} \cdot \overbrace{1}^{g_1^\circ(y)}$$

So then the LHS eigenvalue can be calculated by the trivial case of **Claim 11.6**, and the RHS can be calculated by direct application of star operator

$$\lambda = \left(\sum_{i=0}^{\infty} \frac{f^i(a)}{i!} (x-a)^i \right) \star 1$$

If we believe **Conjecture 11.2**, this RHS can be interpreted as the sum-of-eigenvalues of a specific infinite-dot-product surface.

We therefore have

$$\lambda = \int_0^1 \left(\sum_{i=0}^{\infty} \frac{f^i(a)}{i!} (1-y-a)^i \right) dy$$

[6]One easily checks that the LHS and the RHS are indeed equivalent

$$e - 1 = \sum_{k=1}^{\infty} \frac{1}{k!}$$

by noticing that the Maclaurin expansion of e^x evaluated at $x = 1$ yields

$$e = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}$$

by definition. By absolute convergence of the sum, we can bring in the integral and solve:

$$\begin{aligned} &= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} \left(\int_0^1 (1-y-a)^i \, dy \right) \\ &= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} \left(\frac{-(1-y-a)^{i+1}}{i+1} \bigg|_0^1 \right) \\ &= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!} \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{i+1} \right) \\ &= \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right) \end{aligned}$$

with $a \in [0, 1]$. □

Corollary 11.10. ***(February 23, 2013)*** *The Maclaurin-expandable* $f(x), f\colon [0,1] \times [0,1] \to \mathbb{R}$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^i(0)}{i!} x^i$$

has eigenvalue

$$\lambda = \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!}$$

Proof We have that by **Claim 11.9**, the Taylor-expandable $f(x)$ has eigenvalue

$$\lambda = \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right)$$

with $a \in [0, 1]$. Letting $a = 0$, we get

$$\begin{aligned} \lambda &= \sum_{i=0}^{\infty} f^i(0) \cdot \left(\frac{(1)^{i+1} - (0)^{i+1}}{(i+1)!} \right) \\ &= \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!} \end{aligned}$$

□

Corollary 11.11. ***(February 25, 2013)*** *The Taylor-expansion of* $f(x), f\colon [0,1] \times [0,1] \to \mathbb{R}$ *about* $a = 1$ *has eigenvalue*

$$\lambda = \sum_{i=0}^{\infty} \frac{(-1)^i f^i(1)}{(i+1)!} = \sum_{\substack{i \text{ even} \\ 0 \le i \le \infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(1)}{(i+1)!}$$

Proof This follows directly from **Claim 11.9**, with $f(x)$ has eigenvalue

$$\lambda = \sum_{i=0}^{\infty} f^i(a) \cdot \left(\frac{(1-a)^{i+1} - (-a)^{i+1}}{(i+1)!} \right)$$

At $a = 1$, this becomes

$$\begin{aligned} \lambda &= \sum_{i=0}^{\infty} f^i(1) \cdot \left(\frac{(0)^{i+1} - (-1)^{i+1}}{(i+1)!} \right) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i f^i(1)}{(i+1)!} \\ &= \sum_{\substack{i \text{ even} \\ 0 \le i \le \infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(1)}{(i+1)!} \end{aligned}$$

□

Claim 11.12. ***(February 25, 2013)*** *For any (infinitely-differentiable, Taylor-expandable) function $f(x), f\colon [0,1]\times[0,1]\to\mathbb{R}$,*

$$\sum_{\substack{i\ odd\\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} = \sum_{\substack{i\ even\\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!}$$

Proof Since both **Corollary 11.10** and **Corollary 11.11** are equal to the specific finite-surface eigenvalue, we have the following:

$$\sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!} = \sum_{\substack{i\text{ even}\\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i\text{ odd}\\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!}$$

$$\sum_{\substack{i\text{ even}\\ 0\le i\le\infty}} \frac{f^i(0)}{(i+1)!} + \sum_{\substack{i\text{ odd}\\ 0\le i\le\infty}} \frac{f^i(0)}{(i+1)!} = \sum_{\substack{i\text{ even}\\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!} - \sum_{\substack{i\text{ odd}\\ 0\le i\le\infty}} \frac{f^i(1)}{(i+1)!}$$

$$\sum_{\substack{i\text{ odd}\\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} = \sum_{\substack{i\text{ even}\\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!}$$

□

Remark 11.2. ***(February 25, 2013)*** *It's rather neat that* ***Claim 11.12*** *relates the even and odd derivatives of functions, essentially stating that they are constrained in a very specific way at two particular points.*

Example 11.2. ***(February 26, 2013)*** *Take for example the rather unassuming function $f(x)=(x+1)^2, f\colon [0,1]^2\to\mathbb{R}$. We have:*

$$\begin{array}{ccc} f(x)=(x+1)^2 & f(1)=4 & f(0)=1\\ f'(x)=2(x+1) & f'(1)=4 & f'(0)=2\\ f''(x)=2 & f''(1)=2 & f''(0)=2\\ f'''(x)=0 & f'''(1)=0 & f'''(0)=0\\ \vdots & \vdots & \vdots \end{array}$$

Let's calculate the odd part as in the LHS of ***Claim 11.12****:*

$$\sum_{\substack{i\ odd\\ 0\le i\le\infty}} \frac{f^i(1)+f^i(0)}{(i+1)!} = \frac{4+2}{2!} = 3$$

The even part is:

$$\sum_{\substack{i\ even\\ 0\le i\le\infty}} \frac{f^i(1)-f^i(0)}{(i+1)!} = \frac{4-1}{1!} + \frac{2-2}{3!} = 3$$

And the eigenvalue that gave rise to this invariant is:

$$\lambda = \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!} = \frac{1}{1!} + \frac{2}{2!} + \frac{2}{3!} = 1+1+\frac{1}{3} = 7/3$$

Claim 11.13. ***(April 14, 2013)*** *Let*

$$s^o(n) = \sum_{i=1}^{n} \frac{1}{(2i)!}$$

and

$$s^e(n) = \sum_{i=0}^{n} \frac{1}{(2i+1)!}$$

Then

$$\lim_{n\to\infty} \frac{s^e(n)}{s^o(n)} = \frac{e+1}{e-1} \approx 2.164$$

In other words, the infinite sum $\lim_{n\to\infty} s^e(n)$ is approximately 116% larger than the infinite sum $\lim_{n\to\infty} s^o(n)$.

Proof Another rather interesting example analogous to **Example 11.2** is the function $f(x) = e^x$, with $f\colon [0,1]^2 \to \mathbb{R}$. Here since $f^i(x) = e^x, \forall i \in \mathbb{Z}^+ \cup \{0\}$, we readily evaluate $f^i(1) = e$ and $f^i(0) = 1$ for all non-negative i. Thus we get:

$$\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(1) + f^i(0)}{(i+1)!} = \frac{e+1}{2!} + \frac{e+1}{4!} + \dots$$
$$= (e+1) \cdot \left(\frac{1}{2!} + \frac{1}{4!} + \dots \right)$$

and

$$\sum_{\substack{i \text{ even} \\ 0 \le i \le \infty}} \frac{f^i(1) - f^i(0)}{(i+1)!} = \frac{e-1}{1!} + \frac{e-1}{3!} + \dots$$
$$= (e-1) \cdot \left(\frac{1}{1!} + \frac{1}{3!} + \dots \right)$$

Since these two equations must be equal by **Claim 11.12**, we get

$$\frac{e+1}{e-1} = \frac{\left(\frac{1}{1!} + \frac{1}{3!} + \dots\right)}{\left(\frac{1}{2!} + \frac{1}{4!} + \dots\right)} = \frac{\sum_{i=0}^{\infty} \frac{1}{(2i+1)!}}{\sum_{i=1}^{\infty} \frac{1}{(2i)!}} = \frac{\lim_{n\to\infty} s^e(n)}{\lim_{n\to\infty} s^o(n)} = \lim_{n\to\infty} \frac{s^e(n)}{s^o(n)}$$

□

Claim 11.14. ***(April 14, 2013)*** *The eigenvalue for the function $f(x) = e^x$, with $f\colon [0,1]^2 \to \mathbb{R}$, is*

$$\lambda_{e^x} = e - 1$$

Proof Since, by **Claim 11.10**, we have

$$\lambda_{e^x} = \sum_{i=0}^{\infty} \frac{f^i(0)}{(i+1)!}$$

and $f^i(0) = 1$ for all non-negative i, the substitution yields:

$$\lambda = \sum_{i=0}^{\infty} \frac{1}{(i+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Next, notice that the Maclaurin expansion of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

evaluated at $x = 1$ yeids

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Thus, we get that

$$e - 1 = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \lambda_{e^x}$$

and we are done. □

Claim 11.15. ***(April 14, 2013)*** *Another expression for λ_{e^x} is*

$$\lambda_{e^x} = \lim_{n\to\infty} (s^o(n) + s^e(n)) = \lim_{n\to\infty} s^o(n) + \lim_{n\to\infty} s^e(n)$$

Proof Since

$$\lambda_{e^x} = e - 1 = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

and

$$\lim_{n\to\infty} s^o(n) = \sum_{i=1}^{\infty} \frac{1}{(2i)!} = \frac{1}{2!} + \frac{1}{4!} + \cdots$$

and

$$\lim_{n\to\infty} s^e(n) = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

we can easily see that

$$\lim_{n\to\infty} s^e(n) + \lim_{n\to\infty} s^o(n) = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \lambda_{e^x}$$

as we wanted to show. Thus we have that

$$\lim_{n\to\infty} s^e(n) + \lim_{n\to\infty} s^o(n) = e - 1$$

□

Corollary 11.16. ***(April 14, 2013)*** *The infinite sum*

$$\lim_{n\to\infty} s^o(n) = \sum_{i=1}^{\infty} \frac{1}{(2i)!} = \frac{(e-1)^2}{2e}$$

On the other hand the infinite sum

$$\lim_{n\to\infty} s^e(n) = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} = \frac{e^2-1}{2e}$$

Thus, both infinite sums are convergent.

Proof 1 We have that

$$\lim_{n\to\infty} s^o(n) + \lim_{n\to\infty} s^e(n) = e - 1$$

by **Claim 11.15**. Next, using the ratio

$$\frac{\lim_{n\to\infty} s^e(n)}{\lim_{n\to\infty} s^o(n)} = \frac{e+1}{e-1}$$

from **Claim 11.13**, we get that

$$\lim_{n\to\infty} s^e(n) = \frac{e+1}{e-1} \cdot \lim_{n\to\infty} s^o(n)$$

Thus it must be true that

$$\lim_{n\to\infty} s^o(n) + \frac{e+1}{e-1} \cdot \lim_{n\to\infty} s^o(n) = \left(1 + \frac{e+1}{e-1}\right) \lim_{n\to\infty} s^o(n) = e - 1$$

which, through direct algebraic manipulation yields:

$$\lim_{n\to\infty} s^o(n) = \frac{e-1}{1+\frac{e+1}{e-1}} = \frac{(e-1)\cdot(e-1)}{\left(1+\frac{e+1}{e-1}\right)\cdot(e-1)} = \frac{(e-1)^2}{2e}$$

Next,

$$\begin{aligned}\lim_{n\to\infty} s^e(n) = e - 1 - \lim_{n\to\infty} s^o(n) &= e - 1 - \frac{(e-1)^2}{2e} \\ &= \frac{2e^2 - 2e - \left(e^2 - 2e + 1\right)}{2e} \\ &= \frac{e^2-1}{2e}\end{aligned}$$

□

Proof 2 We can use the Maclaurin expansions of

$$\sinh(x) = \sum_{i=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

and

$$\cosh(x) = \sum_{i=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

evaluated at $x = 1$. Therefore we have

$$\cosh(1) = \sum_{i=0}^{\infty} \frac{1}{(2i)!} = \frac{e^2+1}{2e}$$

Thus

$$\sum_{i=1}^{\infty} \frac{1}{(2i)!} = \frac{e^2+1}{2e} - 1 = \frac{e^2+1}{2e} - \frac{2e}{2e} = \frac{(e-1)^2}{2e}$$

On the other hand we have:

$$\sinh(1) = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} = \frac{e^2 - 1}{2e}$$

□

Claim 11.17. ***(February 26, 2013)*** *For any (infinitely-differentiable, Taylor-expandable) function $f(x), f\colon [0,1] \times [0,1] \to \mathbb{R}$,*

$$\sum_{\substack{i\ odd \\ 0 \le i \le \infty}} \frac{f^i(1) + f^i(0)}{(i+1)!} = \int_0^1 \left(\sum_{\substack{i\ odd \\ 0 \le i \le \infty}} \frac{f^i(x)}{i!} \right) dx$$

Proof An alternate form of **Claim 11.12** follows from shifting the sum indices:

$$\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(1) + f^i(0)}{(i+1)!} = \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^{i+1}(1) - f^{i+1}(0)}{i!}$$

The RHS is, in essence:

$$= \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{\int_0^1 f^i(x)\,dx}{i!}$$

$$= \int_0^1 \left(\sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{f^i(x)}{i!} \right) dx$$

where this last step follows from the fact that sums of subsequences must converge (the odd subsequence), thus allowing us to bring out the integral. □

Claim 11.18. ***(March 14, 2013)*** *The finite polynomial function $f\colon [0,1]^2 \to \mathbb{R}$*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

has eigenvalue

$$\lambda = \sum_{i=0}^{n} \frac{a_i}{i+1}$$

Proof by Induction Take

$$f(x) = a_0$$

with a_0 is a constant. The eigenvalue is

$$\lambda = f(x) \star 1 = \int_0^1 a_0\,dy = a_0$$

which, using the formula, is

$$\lambda = \sum_{i=0}^{0} \frac{a_i}{i+1} = \frac{a_0}{1}$$

and the equivalence is established for the base case. Let's assume the formula works in the kth case. We show the $k+1$th case. So take

$$f(x) = a_{k+1} x^{k+1} + a_k x^k + \ldots + a_1 x + a_0$$

Which has eigenvalue

$$\begin{aligned}\lambda = f(x) \star 1 &= \int_0^1 \Big(a_{k+1}(1-y)^{k+1} + a_k(1-y)^k + \ldots + a_1(1-y) + a_0\Big)\, dy \\ &= \int_0^1 a_{k+1}(1-y)^{k+1}\, dy + \sum_{i=0}^{k} \frac{a_i}{i+1} \\ &= \left.\frac{-a_{k+1}(1-y)^{k+2}}{k+2}\right|_0^1 + \sum_{i=0}^{k} \frac{a_i}{i+1} \\ &= \frac{a_{k+1}}{k+2} + \sum_{i=0}^{k} \frac{a_i}{i+1} \\ &= \sum_{i=0}^{k+1} \frac{a_i}{i+1}\end{aligned}$$

□

Corollary 11.19. ***(March 14, 2013)*** *For finite polynomial functions,*

$$\sum_{i=0}^{n} \frac{f^i(0)}{(i+1)!} = \sum_{i=0}^{n} \frac{a_i}{i+1} = \sum_{i=0}^{n} \frac{(-1)^i f^i(1)}{(i+1)!}$$

Proof This is a consequence of **Claim 11.10**, **Claim 11.11**, and **Claim 11.18**.

Remark 11.3. ***Corollary 11.19*** *essentially relates the constant (last) derivative of each term of a finite polynomial function with the coefficients of such function, and also the (sum of) coefficients of each derivative to the coefficients of the original function.*

Claim 11.20. ***(March 14, 2013)*** *The eigenvalue for the function $f\colon [0,1]^2 \to \mathbb{R}$, $f(x) = a\sin(bx+c)$ for constants a, b, c is*

$$\lambda_{f(x)} = \frac{a}{b}\left(\cos(c) - \cos(b+c)\right)$$

In particular, if $b \in \{(2m+1)\pi\}_{m\in\mathbb{Z}}$ and $c \in \{2\pi n\}_{n\in\mathbb{Z}}$

$$\lambda_{f(x)} = \frac{2a}{(2m+1)\pi}$$

The eigenvalue for the function $g\colon [0,1]^2 \to \mathbb{R}$, $g(x) = a\cos(bx+c)$ is

$$\lambda_{g(x)} = \frac{a}{b}\left(\sin(b+c) - \sin(c)\right)$$

If $b, c \in \{\pi n\}_{n\in\mathbb{Z}}$, then

$$\lambda_{g(x)} = 0$$

Proof For $f(x)$, the eigenvalue is

$$\begin{aligned}\lambda_{f(x)} = f(x) \star 1 = a\int_0^1 \sin(b(1-y)+c)\, dy \\ = a\left.\left(\frac{\cos(b(1-y)+c)}{b}\right)\right|_0^1 \\ = \frac{a}{b}\left(\cos(c) - \cos(b+c)\right)\end{aligned}$$

Choosing $b \in \{(2m+1)\pi\}_{m\in\mathbb{Z}}$ and $c \in \{2\pi n\}_{n\in\mathbb{Z}}$ yields

$$\lambda_{f(x)} = \frac{a}{(2m+1)\pi}(1+1) = \frac{2a}{(2m+1)\pi}$$

Next, take $g(x)$ with eigenvalue

$$\begin{aligned}\lambda_{g(x)} = g(x) \star 1 &= a \int_0^1 \cos(b(1-y)+c)\, dy \\ &= a \left(\frac{\sin(b(1-y)+c)}{-b} \right) \Big|_0^1 \\ &= \frac{a}{b} \left(\sin(b+c) - \sin(c) \right)\end{aligned}$$

It is easy to see that with $b, c \in \{\pi n\}_{n \in \mathbb{Z}}$ we have

$$\lambda_{g(x)} = \frac{a}{b}(0+0) = 0$$

Corollary 11.21. ***(March 16, 2013)***

$$\sum_{\substack{i \ even \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!} = \sum_{\substack{i \ odd \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!}$$

with $n \in \mathbb{Z}$.

Proof Using the Maclaurin expansion of $\cos(\pi n x + \pi n)$, $n \in \mathbb{Z}$ (notice $a = 1, b = \pi n, c = \pi n$), we get:

$$\cos(\pi n x + \pi n) = \sum_{i=0}^{\infty} \frac{(-1)^i (\pi n x + \pi n)^{2i}}{(2i)!}$$

and eigenvalue, by **Claim 11.20**,

$$\begin{aligned}0 &= \int_0^1 \left(\sum_{i=0}^{\infty} \frac{(-1)^i (\pi n(1-y) + \pi n)^{2i}}{(2i)!} \right) dy \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \int_0^1 (\pi n(1-y) + \pi n)^{2i}\, dy \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \left(-\frac{(\pi n(1-y) + \pi n)^{2i+1}}{\pi n(2i+1)} \right) \Big|_0^1 \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \left(\frac{-(\pi n)^{2i+1} + (2\pi n)^{2i+1}}{\pi n(2i+1)} \right) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i (\pi n)^{2i}}{(2i+1)!} \\ 0 &= \sum_{\substack{i \text{ even} \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i}}{(2i+1)!} - \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i}}{(2i+1)!} \\ \sum_{\substack{i \text{ even} \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i}}{(2i+1)!} &= \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i}}{(2i+1)!} \\ \sum_{\substack{i \text{ even} \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!} &= \sum_{\substack{i \text{ odd} \\ 0 \le i \le \infty}} \frac{(\pi n)^{2i+1}}{(2i+1)!}\end{aligned}$$

□

Claim 11.22. ***(April 6, 2013)*** *The Bernoulli polynomials*

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i$$

with $B_n \colon [0,1]^2 \to \mathbb{R}$ *and* b_m *are the Bernoulli numbers, have eigenvalue*

$$\lambda_{B_n(x)} = \sum_{i=0}^{n} \binom{n}{i} \frac{b_{n-i}}{i+1}$$

Proof Begin with

$$\begin{aligned}\lambda_{B_n(x)} &= B_n(x) \star 1\\ &= \int_0^1 B_n(1-y)\,dy\\ &= \int_0^1 \sum_{i=1}^{n} \binom{n}{i} b_{n-i}(1-y)^i\,dy\\ &= \sum_{i=1}^{n} \binom{n}{i} b_{n-i} \int_0^1 (1-y)^i\,dy\\ &= \sum_{i=1}^{n} \binom{n}{i} b_{n-i}\ \frac{-(1-y)^{i+1}}{i+1}\Big|_0^1\\ &= \sum_{i=0}^{n} \binom{n}{i} \frac{b_{n-i}}{i+1}\end{aligned}$$

□

Claim 11.23. ***(April 14, 2013)***

$$\lambda_{B_n(x)} = 0$$

for $n \in \mathbb{Z}^+$ with $b_1 = -\frac{1}{2}$. Notice the explicit exclusion when $n = 0$.

Proof First, for n is odd, we have

$$\int_0^{\frac{1}{2}} \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i\,dx = -\int_{\frac{1}{2}}^1 \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i\,dx$$

This is equal to

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} b_{n-i} \int_0^{\frac{1}{2}} x^i\,dx &= -\sum_{i=0}^{n} \binom{n}{i} b_{n-i} \int_{\frac{1}{2}}^1 x^i\,dx\\ \sum_{i=0}^{n} \binom{n}{i} b_{n-i}\ \frac{x^{i+1}}{i+1}\Big|_0^{\frac{1}{2}} &= -\sum_{i=0}^{n} \binom{n}{i} b_{n-i}\ \frac{x^{i+1}}{i+1}\Big|_{\frac{1}{2}}^1\\ \sum_{i=0}^{n} \binom{n}{i} b_{n-i} \frac{\left(\frac{1}{2}\right)^{i+1}}{i+1} &= -\sum_{i=0}^{n} \binom{n}{i} b_{n-i} \frac{1}{i+1} + \sum_{i=0}^{n} \binom{n}{i} b_{n-i} \frac{\left(\frac{1}{2}\right)^{i+1}}{i+1}\\ \sum_{i=0}^{n} \binom{n}{i} \frac{b_{n-i}}{i+1} &= 0\end{aligned}$$

Next, if n is even, $n \neq 0$, □

Claim 11.24. ***(April 16, 2013)*** *Bernoulli polynomials plus one, $n > 0$, are* Pasquali patches*:*

$$B_n(x) + 1 = p^n(x)$$

11.5. **Specific Infinite-Dot-Product Surfaces.**

11.6. **Relevant Generalizations as Applied to *Pasquali Patches*.**

Let's take what we have learned and refocus on *Pasquali patches.*

Claim 11.25. ***(March 20, 2016)*** *Take **Construction 2.1**. We show that $\lambda_{p(x,y),1} = 1$ (thus all* Pasquali patches *constructed as by **Construction 2.1** have an eigenvalue equal to 1, a remarkable fact) and $\lambda_{p(x,y),2} = \big(f_1(x) - F_1 \cdot f_2^*(x)\big) \star g_1(y)$, with $f_2^*(x) = \frac{f_2(x)}{F_2}$ and $F_2^* = 1$.*

Proof The form of **Construction 2.1** is

$$p(x,y) = f_1(x)g_1(y) + f_2(x)\frac{1 - g_1(y)F_1}{F_2}$$

There are only two eigenvalues, λ_1 and λ_2 because this is a two-dimensional Specific Finite-Dot-Product Surface. Hence we know from **Claim 11.5** that

$$\lambda_{1,2} = \frac{C_{2,2} + C_{1,1} \pm \sqrt{(C_{2,2} - C_{1,1})^2 + 4C_{1,2}C_{2,1}}}{2}$$

From this equation we have the following quantities:

$$\begin{aligned} C_{1,1} + C_{2,2} &= f_1(x) \star g_1(y) + f_2(x) \star \left(\tfrac{1-g_1(y)F_1}{F_2}\right) \\ &= f_1(x) \star g_1(y) - f_2^*(x) \star g_1(y)F_1 + 1 \\ &= 1 + f_1(x) \star g_1(y) - \underbrace{f_2^*(x) \star g_1(y)F_1}_{L} \end{aligned}$$

Next,

$$\begin{aligned} (C_{2,2} - C_{1,1})^2 &= \left(f_2(x) \star \left(\tfrac{1-g_1(y)F_1}{F_2}\right) - f_1(x) \star g_1(y)\right)^2 \\ &= \big(1 - f_2^*(x) \star g_1(y)F_1 - f_1(x) \star g_1(y)\big)^2 \\ &= \left(\underbrace{(1 - f_1(x) \star g_1(y))}_{K} - \underbrace{f_2^*(x) \star g_1(y)F_1}_{L}\right)^2 \end{aligned}$$

Finally we have

$$\begin{aligned} C_{1,2} \cdot C_{2,1} &= f_1(x) \star \tfrac{1-g_1(y)F_1}{F_2} \cdot f_2(x) \star g_1(y) \\ &= (1 - f_1(x) \star g_1(y))\, \tfrac{F_1}{F_2} \cdot f_2(x) \star g_1(y) \\ &= \underbrace{(1 - f_1(x) \star g_1(y))}_{K} \cdot \underbrace{f_2^*(x) \star g_1(y)F_1}_{L} \end{aligned}$$

With the help of the expression simplifications, we have

$$\begin{aligned} \lambda_{1,2} &= \tfrac{1+f_1(x)\star g_1(y)-L\pm\sqrt{(K-L)^2+4\cdot K\cdot L}}{2} \\ &= \tfrac{1+f_1(x)\star g_1(y)-L\pm\sqrt{\left(K^2-2\cdot K\cdot L+L^2\right)+4\cdot K\cdot L}}{2} \\ &= \tfrac{1+f_1(x)\star g_1(y)-L\pm\sqrt{(K+L)^2}}{2} \\ &= \tfrac{1+f_1(x)\star g_1(y)-L\pm(K+L)}{2} \end{aligned}$$

The first eigenvalue is:

$$\begin{aligned} \lambda_1 &= \tfrac{1+f_1(x)\star g_1(y)-L+(K+L)}{2} \\ &= \tfrac{1+f_1(x)\star g_1(y)+K}{2} \\ &= \tfrac{1+\cancel{f_1(x)\star g_1(y)}+\left(1-\cancel{f_1(x)\star g_1(y)}\right)}{2} \\ &= 1 \end{aligned}$$

The second eigenvalue is:

$$\begin{aligned} \lambda_2 &= \tfrac{1+f_1(x)\star g_1(y)-L-(K+L)}{2} \\ &= \tfrac{1+f_1(x)\star g_1(y)-K-2L}{2} \\ &= \frac{\cancel{1}+f_1(x)\star g_1(y)-\left(\cancel{1}-f_1(x)\star g_1(y)\right)-\cancel{2}\cdot f_2^*(x)\star g_1(y)F_1}{\cancel{2}} \\ &= f_1(x) \star g_1(y) - f_2^*(x) \star g_1(y)F_1 \\ &= \big(f_1(x) - F_1 \cdot f_2^*(x)\big) \star g_1(y) \end{aligned}$$

□

Corollary 11.26. ***(March 20, 2016)*** *It follows that the eigenvalues for* ***Construction 2.3*** *are $\lambda_1 = 1$ and $\lambda_2 = f_1(x) \star g_1(y)$.*

Proof By **Claim 11.25** we have that $\lambda_{p(x,y),1} = 1$ and $\lambda_{p(x,y),2} = \big(f_1(x) - F_1 \cdot f_2^*(x)\big) \star g_1(y)$. **Construction 2.3** is a restriction of **Construction 2.1** with $F_1 = 0$. Substitution yields the desired result. □

Claim 11.27. ***(March 30, 2016)*** *For the generalization of* ***Construction 2.1****, as delineated in* ***Construction 2.2****, that is:*

$$p(x, y) = (\mathbf{f}_{n-1}(x) - f_n^*(x) \cdot \mathbf{F}_{n-1}) \cdot \mathbf{g}_{n-1}(y) + f_n^*(x)$$

we can also assert the fact that one of the n eigenvalues is $\lambda_1 = 1$.

Proof We will use the normalized definition as stated to create the characteristic eigenvalue equation. Observe the following redefinition for constants $C_{i,j}$ using the normalized form:

$$p(x,y) = \overbrace{(\mathbf{f}_{n-1}(x) - f_n^*(x)\cdot\mathbf{F}_{n-1})}^{f^\circ_{n-1}(x)} \cdot \overbrace{\mathbf{g}_{n-1}(y)}^{g^\circ_{n-1}(y)} + \overbrace{f_n^*(x)}^{f^\circ_n(x)} \cdot \overbrace{1}^{g^\circ_n(y)}$$

Next, recall that $C_{i,j} = f_i^\circ(x) \star g_j^\circ(y)$ and

$$\begin{array}{ccc} C_{1,1} & = & (f_1(x) - f_n^*(x)\cdot F_1)\star g_1(y) \\ C_{2,2} & = & (f_2(x) - f_n^*(x)\cdot F_2)\star g_2(y) \\ \vdots & & \vdots \\ C_{n-1,n-1} & = & (f_{n-1}(x) - f_n^*(x)\cdot F_{n-1})\star g_{n-1}(y) \\ C_{n,n} & = & f_n^*(x)\star g_n(y) = f_n^*(x)\star 1 = \int_0^1 f_n^*(1-y)\,dy = 1 \end{array}$$

and

$$\begin{array}{ccc} C_{k,n} & = & (f_k(x) - f_n^*(x)\cdot F_k)\star g_n(y) \\ & = & (f_k(x) - f_n^*(x)\cdot F_k)\star 1 \\ & = & \int_0^1 (f_k(1-y) - f_n^*(1-y)\cdot F_k)\ dy \\ & = & F_k - 1\cdot F_k \\ & = & 0 \end{array}$$

This is enough detail to build the characteristic matrix as in **Claim 11.6** that will prove our claim:

$$\det\begin{vmatrix} (f_1(x) - f_n^*(x)\cdot F_1)\star g_1(y) - \lambda & C_{2,1} & \dots & C_{n-1,1} & C_{n,1} \\ C_{1,2} & (f_2(x) - f_n^*(x)\cdot F_2)\star g_2(y) - \lambda & \dots & C_{n-1,2} & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,n-1} & C_{2,n-1} & \dots & C_{n-1,n-1} - \lambda & C_{n,n-1} \\ 0 & 0 & \dots & 0 & 1-\lambda \end{vmatrix} = 0$$

This is very lucky, because Laplace's formula allows us to define the determinant equation using minors by focusing on, say, the (final, nth) row of zeros, which results in all minor cancellations except:

$$(1-\lambda)\cdot|M_{n,n}| = 0$$

Clearly, the characteristic polynomial function on λ contains $\lambda = 1$ as a zero, and we are done. □

Corollary 11.28. ***(March 30, 2016)*** *The restriction of* ***Construction 2.2****, that is,* ***Construction 2.4****, also has* $\lambda = 1$ *as an eigenvalue.*

Proof This follows from the fact that **Construction 2.4** is a restriction of **Construction 2.2**, but can also be seen at the determinant, since

$$\begin{array}{ccc} C_{1,1} & = & \left(f_1(x) - \cancel{f_n^*(x)\cdot F_1}\right)\star g_1(y) = f_1(x)\star g_1(y) \\ C_{2,2} & = & \left(f_2(x) - \cancel{f_n^*(x)\cdot F_2}\right)\star g_2(y) = f_2(x)\star g_2(y) \\ \vdots & & \vdots \\ C_{n-1,n-1} & = & (f_{n-1}(x) - \cancel{f_n^*(x)\cdot F_{n-1}})\star g_{n-1}(y) = f_{n-1}(x)\star g_{n-1}(y) \\ C_{n,n} & = & f_n^*(x)\star g_n(y) = f_n^*(x)\star 1 = \int_0^1 f_n^*(1-y)\,dy = 1 \end{array}$$

and

$$\begin{array}{ccc} C_{k,n} & = & \left(f_k(x) - \cancel{f_n^*(x)\cdot F_k}\right)\star g_n(y) \\ & = & f_k(x)\star 1 \\ & = & \int_0^1 f_k(1-y)\,dy \\ & = & F_k \\ & = & 0 \end{array}$$

with $F_k = 0$, $k \in \{1,\dots,n-1\}$ as per the construction premise. This last bit of course results in the last row of the determinant of the characteristic matrix being

$$\left[\begin{array}{cccc} 0 & 0 & \dots & 1-\lambda \end{array}\right]$$

and the equation on minors using Laplace's formula yields, as before,

$$(1-\lambda)\cdot|M_{n,n}| = 0$$

Again $\lambda = 1$ for all such constructions. □

Claim 11.29. ***(April 9, 2016)*** Pasquali patches *constructed as by* ***Construction 2.2*** *are closed.*

Proof To make the proof clear, let us relabel the normalized version of **Construction 2.2** as

$$p(x,y) = \overbrace{(\mathbf{f}_{n-1}(x) - f_n^*(x)\cdot \mathbf{F}_{n-1})}^{\mathbf{P}_x(x)} \cdot \overbrace{\mathbf{g}_{n-1}(y)}^{\mathbf{P}_y(y)} + f_n^*(x)$$

so as to manipulate the equation more simply, and

$$q(x,y) = \mathbf{Q}_x(x)\cdot \mathbf{Q}_y(y) + h_n^*(x)$$

with $F_n^* = 1$ and $H_n^* = 1$. Then

$$\begin{aligned} p(x,y)\star q(x,y) &= j\left(\int_0^1 (\mathbf{P}_x(1-y)\cdot \mathbf{P}_y(t) + f_n^*(1-y))\cdot(\mathbf{Q}_x(x)\cdot\mathbf{Q}_y(y) + h_n^*(x))\,dy\right)\\ &= [\alpha\cdot \mathbf{Q}_x(x) + \cancel{\beta\cdot h_n^*(x)}]\cdot \mathbf{P}_y(y) + \gamma\cdot \mathbf{Q}_x(x)\cdot \mathbf{1} + h_n^*(x)\end{aligned}$$

with $\alpha = \int_0^1 \mathbf{P}_x(1-y)\cdot \mathbf{Q}_y(y)\,dy$, $\beta = \int_0^1 \mathbf{P}_x(1-y)\cdot \mathbf{1}\,dy = 0$, and $\gamma = \int_0^1 f_n^*(1-y)\cdot \mathbf{Q}_y(y)\,dy$. We're not too concerned of the *form* of the resultant star product as much as its *structure*. Observe

$$r(x,y) = \overbrace{\alpha\cdot \mathbf{Q}_x(x)}^{\mathbf{R}_x^a(x)}\cdot \overbrace{\mathbf{P}_y(y)}^{\mathbf{R}_y^a(y)} + \overbrace{\gamma\cdot \mathbf{Q}_x(x)}^{\mathbf{R}_x^b(x)}\cdot \overbrace{\mathbf{1}}^{\mathbf{R}_y^b(y)} + h_n^*(x)$$

can be folded back into function vectors $\mathbf{R}_x(x)$ and $\mathbf{R}_y(y)$. Thus the structure of **Construction 2.2** functions is preserved when we multiply one by another, showing closure. Of course the property of **Construction 2.2** being *Pasquali patches* means $r(x,y)$ is closed under *that* property, and so is a *Pasquali patch* also, as can be seen when we integrate across x:

$$\int_0^1 r(x,y)\,dx = \int_0^1 \cancel{\alpha\cdot \mathbf{Q}_x(x)\cdot \mathbf{P}_y(y)} + \cancel{\gamma\cdot \mathbf{Q}_x(x)\cdot \mathbf{1}} + h_n^*(x)\,dx = 1$$

and the unit contribution is given by $h_n^*(x)$. □

Claim 11.30. ***(April 10, 2016)*** Pasquali patches *constructed as by* ***Construction 2.2*** *have powers:*

$$p_n(x,y) = \alpha^{n-1}\cdot \mathbf{P}_x(x)\cdot \mathbf{P}_y(y) + \sum_{i=0}^{n-2}\alpha^i\cdot\gamma\cdot \mathbf{P}_x(x)\cdot \mathbf{1} + f_n^*(x)$$

with

$$\alpha = \int_0^1 \mathbf{P}_x(1-y)\cdot \mathbf{P}_y(y)\,dy = \mathbf{P}_x(x)\star \mathbf{P}_y(y) = \operatorname{str}[\mathbf{P}_x(x)\cdot \mathbf{P}_y(y)]$$

and

$$\gamma = \int_0^1 f_n^*(1-y)\cdot \mathbf{P}_y(y)\,dy = f_n^*(x)\star \mathbf{P}_y(y)$$

Proof by Induction First, using the formula observe

$$p(x,y) = \alpha^0\cdot \mathbf{P}_x(x)\cdot \mathbf{P}_y(y) + f_n(x)$$

and the second power

$$p_2(x,y) = \alpha\cdot \mathbf{P}_x(x)\cdot \mathbf{P}_y(y) + \gamma\cdot \mathbf{P}_x(x) + f_n(x)$$

which is exactly what we expect from the definition of **Construction 2.2** and **Claim 11.29**. Next, let us assume that the formula works and

$$p_k(x,y) = \alpha^{k-1}\cdot \mathbf{P}_x(x)\cdot \mathbf{P}_y(y) + \sum_{i=0}^{k-2}\alpha^i\cdot\gamma\cdot \mathbf{P}_x(x)\cdot \mathbf{1} + f_n^*(x)$$

Let us examine $p_{k+1}(x,y) = p_k(x,y)\star p(x,y)$

$$p_{k+1}(x,y) = j\left(\int_0^1\left(\alpha^{k-1}\cdot \mathbf{P}_x(1-y)\cdot \mathbf{P}_y(t) + \sum_{i=0}^{k-2}\alpha^i\cdot\gamma\cdot \mathbf{P}_x(1-y)\cdot \mathbf{1} + f_n^*(1-y)\right)\cdot(\mathbf{P}_x(x)\cdot \mathbf{P}_y(y) + f_n^*(x))\,dy\right)$$

term by term upon dotting. The first dot the first term is:

$$\alpha^{k-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) \cdot \underbrace{\int_0^1 \mathbf{P}_x(1-y) \cdot \mathbf{P}_y(y)\, dy}_{\alpha} = \alpha^k \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y)$$

The first dot the last term is:

$$\alpha^{k-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) \cdot f_n^*(x) \cdot \underbrace{\int_0^1 \mathbf{P}_x(1-y) \cdot \mathbf{1}\, dy}_{\beta} = \cancel{\alpha^{k-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) \cdot f_n^*(x) \cdot 0}$$

The last dot the first term is:

$$\mathbf{P}_x(x) \cdot \int_0^1 f_n^*(1-y) \cdot \mathbf{P}_y(y)\, dy = \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1}$$

The last dot the last term is:

$$f_n^*(x) \cdot \int_0^1 f_n^*(1-y)\, dy = f_n^*(x)$$

The middle dot the first term is:

$$\sum_{i=0}^{k-2} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \underbrace{\int_0^1 \mathbf{P}_x(1-y) \cdot \mathbf{P}_y(y)\, dy}_{\alpha} = \sum_{i=0}^{k-2} \alpha^{i+1} \cdot C \cdot \mathbf{P}_x(x) \cdot \mathbf{1}$$

Finally, the middle dot the last term vanishes:

$$\sum_{i=0}^{k-2} \alpha^i \cdot \gamma \cdot f_n^*(x) \cdot \underbrace{\int_0^1 \mathbf{P}_x(1-y) \cdot \mathbf{1}\, dy}_{\beta} = \cancel{\sum_{i=0}^{k-2} \alpha^i \cdot \gamma \cdot f_n^*(x) \cdot 0}$$

Putting all this information together we get:

$$\begin{aligned} p_{k+1}(x,y) &= \alpha^k \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + \textstyle\sum_{i=0}^{k-2} \alpha^{i+1} \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x) \\ &= \alpha^k \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \left(\textstyle\sum_{i=0}^{k-2} \alpha^{i+1} + 1\right) \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x) \\ &= \alpha^k \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \textstyle\sum_{i=0}^{k-1} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x) \end{aligned}$$

□

Corollary 11.31. ***(April 10, 2016)*** *It follows that*

$$p_\infty(x) = \frac{\gamma}{1-\alpha} \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

and $|\alpha| < 1$, γ bounded, both conditions needed to establish that such a limiting surface indeed exists (convergence criterion). Furthermore, we check that this is indeed a Pasquali patch.

Proof By **Definition 2.2**,

$$p_\infty(x) = \lim_{n\to\infty} p_n(x,y) = \lim_{n\to\infty} \left[\alpha^{n-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \sum_{i=0}^{n-2} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)\right]$$

Next, by **Claim 2.12**, the functions of y must vanish at the limit. Thus, it follows that $|\alpha| < 1$. We have now established bounds on α, which happen to be exactly the radius of convergence of the geometric series:

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$$

and

$$p_\infty(x) = \cancel{\lim_{n\to\infty} \left[\alpha^{n-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y)\right]} + \lim_{n\to\infty} \left[\sum_{i=0}^{n-2} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1}\right] + f_n^*(x)$$

gives the desired result:

$$p_\infty(x) = \frac{\gamma}{1-\alpha} \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

As a check, we integrate across x to corroborate the definition of *Pasquali patch*:

$$\int_0^1 p_\infty(x)\,dx = \frac{\gamma}{1-\alpha} \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{1}\,dx + \int_0^1 f_n^*(x)\,dx = 1$$

□

Corollary 11.32. ***(April 10, 2016)*** Pasquali patches *constructed as by* ***Construction 2.2*** *retain the* Pasquali patch $f_n^*(x)$*, with* $F_n^* = 1$*, in all star powers (including the infinite power).*

Proof This follows from **Claim 11.30** and **Corollary 11.31**. □

Claim 11.33. ***(April 11, 2016)***

$$p_\infty(x) = \frac{\gamma}{1-\alpha} \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

is the eigenfunction corresponding to eigenvalue $\lambda = 1$ *of all* ***Construction 2.2*** *functions, through each power independently.*

Proof An eigenfunction $e(x)$ has the property

$$e(x) \star h(x,y) = \lambda e(x)$$

where the eigenfunction's corresponding eigenvalue is λ. The claim is more ambitious, and we will show that $p_\infty(x) \star p_n(x,y) = 1 \cdot p_\infty(x)$ for any $n \in \mathbb{Z}^+$. The left-hand side is

$$j\left(\int_0^1 \left(\frac{\gamma}{1-\alpha} \cdot \mathbf{P}_x(1-y) \cdot \mathbf{1} + f_n^*(1-y)\right) \cdot \left(\alpha^{n-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \sum_{i=0}^{n-2} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)\right) dy\right)$$

Observe the first term dotted with the middle and last term produce β which annihilates the results, so that the only relevant term is the first dot the first:

$$\frac{\gamma}{1-\alpha} \cdot \alpha^{n-1} \cdot \mathbf{P}_x(x) \cdot \underbrace{\int_0^1 \mathbf{P}_x(1-y) \cdot \mathbf{P}_y(y)\,dy}_{\alpha} = \frac{\gamma}{1-\alpha} \cdot \alpha^n \cdot \mathbf{P}_x(x) \cdot \mathbf{1}$$

The second term dot the first produces:

$$\alpha^{n-1} \cdot \mathbf{P}_x(x) \cdot \underbrace{\int_0^1 f_n^*(1-y) \cdot \mathbf{P}_y(y)\,dy}_{\gamma} = \alpha^{n-1} \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1}$$

The second term and the second:

$$\sum_{i=0}^{n-1} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} \cdot \int_0^1 f_n^*(1-y)\,dy = \sum_{i=0}^{n-1} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1}$$

and the second by the last term gives

$$\int_0^1 f_n^*(1-y) \cdot f_n^*(x)\,dy = f_n^*(x)$$

Factoring gives

$$\left(\frac{\alpha^n}{1-\alpha} + \alpha^{n-1} + \sum_{i=0}^{n-2} \alpha^i\right) \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x) = \left(\frac{\alpha^n}{1-\alpha} + \sum_{i=0}^{n-1} \alpha^i\right) \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

The parenthetical part of this last formulation is equivalent to

$$\begin{aligned} \tfrac{\alpha^n}{1-\alpha} + \tfrac{1-\alpha}{1-\alpha} \cdot \textstyle\sum_{i=0}^{n-1} \alpha^i &= \tfrac{1}{1-\alpha}\left(\alpha^n + \textstyle\sum_{i=0}^{n-1} \alpha^i - \textstyle\sum_{i=0}^{n-1} \alpha^{i+1}\right) \\ &= \tfrac{1}{1-\alpha}\left(\textstyle\sum_{i=0}^{n} \alpha^i - \textstyle\sum_{i=1}^{n} \alpha^i\right) \\ &= \tfrac{1}{1-\alpha} \end{aligned}$$

within the bounds already established for α, and the result of the star product is

$$\frac{\gamma}{1-\alpha} \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x) = p_\infty(x)$$

as we wanted to show. □

Lemma 11.34. ***(April 15, 2016)*** *$\mathbf{P}_x(x) \cdot \mathbf{1} = 0$ if and only if $\mathbf{P}_x(x)$ is of even cardinality and half the functions are opposite the other half.*

Proof We have that:

$\Rightarrow$ Suppose that $\mathbf{P}_x(x)$ is a single function $u_1(x)$ so that the expression $\mathbf{P}_x(x) \cdot \mathbf{1} = 0$ implies $u_1(x) = 0$. This is clearly a contradiction, since $\mathbf{P}_x(x)$ is of nonzero cardinality. Suppose the expression

$$\sum_{i=1}^{2k} u_i(x) + u_{2k+1}(x) = 0$$

likewise derives a contradiction, so that

$$\sum_{i=1}^{2k} u_i(x) + u_{2k+1}(x) \neq 0$$

and in fact equals some function, say, $v(x) \neq 0$. Look at

$$\sum_{i=1}^{2(k+1)} u_i(x) + u_{2(k+1)+1}(x) = 0 \Rightarrow \underbrace{\sum_{i=1}^{2k} u_i(x) + u_{2k+1}(x)}_{v(x)} + u_{2k+2}(x) + u_{2k+3}(x) = 0$$

For the expression to be true, all functions must be zero, but which contradicts the fact that $\mathbf{P}_x(x)$ is a nonzero vector (and that $v(x) \neq 0$). If two functions are opposites, say the last two, and cancel each other out, again this implies $v(x) = 0$, contrary to our assumption. We have exhausted the possible scenarios. By inducing, odd cardinality contradicts the expression $\mathbf{P}_x(x) \cdot \mathbf{1} = 0$, and so $\mathbf{P}_x(x)$ must be even cardinality with n functions opposite other n functions.

$\Leftarrow$

$$\mathbf{P}_x(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \\ u_{n+1}(x) \\ \vdots \\ u_{2n-1}(x) \\ u_{2n}(x) \end{bmatrix} = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \\ -u_n(x) \\ \vdots \\ -u_2(x) \\ -u_1(x) \end{bmatrix}$$

then

$$\mathbf{P}_x(x) \cdot \mathbf{1} = \sum_{i=1}^{2n} u_i(x) = \sum_{i=1}^{n} u_i(x) + \sum_{i=n+1}^{2n} -u_i(x) = \sum_{i=1}^{n} u_i(x) - \sum_{i=1}^{n} u_i(x) = 0$$

□

11.6.1. *Particularities of this* Pasqualian.

Claim 11.35 (*Pasqualian* Broad Existence Criteria)**.** ***(July 11, 2016)*** *If a* Pasqualian *exists, then* $0 \leq \alpha < 1$.

Proof by Contrapositive We show that if $\alpha < 0$ or $\alpha > 1$ or $\alpha = 1$, the *Pasqualian* does not exist. With

$$p_n(x, y) = \alpha^{n-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \sum_{i=0}^{n-2} \alpha^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

take $\alpha < 0$, as by $\alpha = (-\eta)$. Then

$$p_n(x, y) = (-\eta)^{n-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + \sum_{i=0}^{n-2} (-\eta)^i \cdot \gamma \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

Consider $(-\eta)^{n-1}$. In particular, n must be an integer in order for the power to be defined, and partial powers cannot be defined because the negative sign forbids it (at least not in the real realm). Thus we cannot force n on $\mathbb{R}$, and a smooth function of t cannot be created. A *Pasqualian* cannot exist.

Next, consider $\alpha > 1$. Observe that

$$\lim_{n\to\infty} \alpha^{n-1} = \infty$$

A *Pasqualian* is such that

$$\lim_{n\to\infty} p_n(x,y) = \lim_{t\to\infty} p(x,y,t) = p_\infty(x)$$

is a limiting surface and is well defined. But with $\alpha > 1$,

$$\lim_{n\to\infty} p_n(x,y) = \lim_{t\to\infty} p(x,y,t) = \infty$$

and a *Pasqualian* cannot exist.

Finally, let $\alpha = 1$, and

$$\begin{aligned} p_n(x,y) &= \mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + \textstyle\sum_{i=0}^{n-2}\gamma\cdot\mathbf{P}_x(x)\cdot\mathbf{1} + f_n^*(x) \\ &= \mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + (n-1)\cdot\gamma\cdot\mathbf{P}_x(x)\cdot\mathbf{1} + f_n^*(x) \end{aligned}$$

but it is clear that $p_n(x,y)$ diverges unboundedly at all places not equal to zero in $x, y \in [0,1]$. Again, there can be no *Pasqualian*.

□

Claim 11.36. ***(April 16, 2016)*** *If $\alpha = 0$, then there exists an induced* Pasqualian *and it is time-independent.*

Proof $\alpha = 0$ means the time dependence in the induced *Pasqualian* vanishes:

$$p_n(x,y) = \cancel{\alpha^{n-1}\cdot\mathbf{P}_x(x)\cdot\mathbf{P}_y(y)} + \cancel{\sum_{i=0}^{n-2}\alpha^i\cdot\gamma\cdot\mathbf{P}_x(x)\cdot\mathbf{1}} + f_n^*(x) = p(x)$$

Therefore the *Pasqualian* is

$$p(x,y,t) = f_n^*(x)$$

□

Claim 11.37 (*Pasqualian* Strict Existence Criteria)**.** ***(July 11, 2016)***

Symbolically,

$$\exists Pasqualian \iff 0 \le \alpha \le 1 \wedge [\gamma = 0 \vee \mathbf{P}_x(x)\cdot\mathbf{1} = 0]$$

Therefore, the form *of the* Pasqualian *is:*

$$p(x,y,t) = \alpha^{t-1}\cdot\mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + f_n^*(x)$$

with $0 \le \alpha \le 1$.

Proof We have:

$\Rightarrow$ The contrapositive is

$$\overbrace{[\alpha < 0 \vee \alpha > 1]}^{\mathbf{A}} \vee \overbrace{[\gamma \ne 0 \wedge \mathbf{P}_x(x) \ne 0]}^{\mathbf{B}} \Rightarrow \nexists Pasqualian$$

Condition **A** has already been shown before. Let us show the implication using Condition **B**. Recall that

$$p_n(x,y) = \alpha^{n-1}\cdot\mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + \sum_{i=0}^{n-2}\alpha^i\cdot\gamma\cdot\mathbf{P}_x(x)\cdot\mathbf{1} + f_n^*(x)$$

Since both γ and $\mathbf{P}_x(x)\cdot\mathbf{1}$ are nonzero, the middle term cannot vanish. The sum is integer-dependent, and so we cannot do the sum over non-integer segments. A *Pasqualian* cannot exist.

$\Leftarrow$ Since $\gamma = 0 \vee \mathbf{P}_x(x)\cdot\mathbf{1} = 0$, it follows that

$$p_n(x,y) = \alpha^{n-1}\cdot\mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + \cancel{\sum_{i=0}^{n-2}\alpha^i\cdot\gamma\cdot\mathbf{P}_x(x)\cdot\mathbf{1}} + f_n^*(x)$$

so we can posit the *Pasqualian*

$$p(x,y,t) = \alpha^{t-1}\cdot\mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + f_n^*(x)$$

Take $\alpha = 0$ and we have the time-independent case $p(x,y,t) = f_n^*(x) = p(x)$. Take $0 < \alpha < 1$ and

$$\lim_{t\to\infty} p(x,y,t) = f_n^*(x)$$

exists and is consistent with the definition of the *Pasqualian*. Finally, take $\alpha = 1$ and

$$p(x,y,t) = \mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + f_n^*(x)$$

with

$$\lim_{t\to\infty} p(x,y,t) = \mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + f_n^*(x)$$

and again we have consistency in terms of definition: the derivative, for example, is zero everywhere and in particular at the limit as $t\to\infty$.

□

Lemma 11.38. ***(July 12, 2016)*** *We may calculate the constant* $\kappa = p_\infty(x)\star\mathbf{P}_y$ *by*

$$\kappa = \frac{\gamma}{1-\alpha}$$

provided $|\alpha|<1$.

Proof We have that:

$$\begin{aligned}
\kappa = p_\infty(x)\star\mathbf{P}_y &= \left(\tfrac{\gamma}{1-\alpha}\cdot\mathbf{P}_x(x)\cdot\mathbf{1} + f_n^*(x)\right)\star\mathbf{P}_y(y)\\
&= \tfrac{\gamma}{1-\alpha}\cdot\overbrace{\mathbf{P}_x(x)\star\mathbf{P}_y(y)}^{\alpha} + \overbrace{f_n^*(x)\star\mathbf{P}_y(y)}^{\gamma}\\
&= \left(\tfrac{\alpha}{1-\alpha}+1\right)\cdot\gamma\\
&= \tfrac{\gamma}{1-\alpha}
\end{aligned}$$

□

Corollary 11.39. ***(July 13, 2016)*** κ *is periodic in the following sense:*

$$\kappa = \alpha\cdot\kappa + \gamma$$

Proof This follows from the definition of κ:

$$\begin{aligned}
\kappa &= \tfrac{\gamma}{1-\alpha}\\
\kappa\cdot(1-\alpha) &= \kappa\\
\kappa &= \alpha\cdot\kappa+\gamma
\end{aligned}$$

Example 11.3. ***(July 11, 2016)*** *Recall* ***Example 5.5*** *with specification* $f_1(x) = x^2$, $f_2(x) = 2x$, $g_1(y) = y^3$, $B = \frac{6}{61}$, *so that*

$$p(x,y) = x^2y^3 + 2x\left(1-\frac{y^3}{3}\right)$$

Next, recall that

$$\begin{aligned}
\mathbf{P}_x(x) &= \mathbf{f}_{n-1}(x) - f_n^*(x)\cdot\mathbf{F}_{n-1}\\
&= f_1(x) - f_2^*(x)\cdot\textstyle\int_0^1 f_1(x)\,dx\\
&= x^2 - 2x\cdot\tfrac{1}{3}\\
&= x^2 - \tfrac{2x}{3}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}_y(y) &= \mathbf{g}_{n-1}(y)\\
&= g_1(y)\\
&= y^3
\end{aligned}$$

Thus we have

$$\alpha = \mathbf{P}_x(x)\star\mathbf{P}_y(y) = \int_0^1\left((1-y)^2 - \frac{2\,(1-y)}{3}\right)\cdot y^3\,dy = -\frac{1}{60}$$

and

$$\gamma = f_n^*(x)\star\mathbf{P}_y(y) = \int_0^1 2(1-y)\cdot y^3\,dy = \frac{1}{10}$$

First, notice that a Pasqualian *cannot exist since* $\alpha<0$, *a simple check shows. Nevertheless, we can obtain the* Pasquali patch *powers via the equation*

$$p_n(x,y) = \alpha^{n-1}\cdot\mathbf{P}_x(x)\cdot\mathbf{P}_y(y) + \sum_{i=0}^{n-2}\alpha^i\cdot\gamma\cdot\mathbf{P}_x(x)\cdot\mathbf{1} + f_n^*(x)$$

so that, for example,

$$\begin{aligned}
p_2(x,y) &= \left(-\tfrac{1}{60}\right)\cdot\left(x^2-\tfrac{2x}{3}\right)\cdot\left(y^3\right) + \tfrac{1}{10}\cdot\left(x^2-\tfrac{2x}{3}\right)+2x\\
&= \tfrac{29x}{15} + \tfrac{y^3x}{90} + \tfrac{x^2}{10} - \tfrac{y^3x^2}{60}
\end{aligned}$$

or

$$\begin{aligned} p_3(x,y) &= \left(-\tfrac{1}{60}\right)^2 \cdot \left(x^2 - \tfrac{2x}{3}\right) \cdot \left(y^3\right) + \tfrac{1}{10} \cdot \left(x^2 - \tfrac{2x}{3}\right) + \left(-\tfrac{1}{60}\right) \cdot \tfrac{1}{10} \cdot \left(x^2 - \tfrac{2x}{3}\right) + 2x \\ &= \left(\left(-\tfrac{1}{60}\right)^2 \cdot y^3 + \tfrac{1}{10} + \left(-\tfrac{1}{60}\right) \cdot \tfrac{1}{10}\right) \cdot \left(x^2 - \tfrac{2x}{3}\right) + 2x \\ &= \tfrac{1741x}{900} - \tfrac{y^3 x}{5400} + \tfrac{59x^2}{600} + \tfrac{y^3 x^2}{3600} \end{aligned}$$

Since $|\alpha| < 1$, *we are assured steady-state and that*

$$\begin{aligned} p_\infty(x) &= \tfrac{\gamma}{1-\alpha} \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x) \\ &= \tfrac{\frac{1}{10}}{1+\frac{1}{60}} \cdot \left(x^2 - \tfrac{2x}{3}\right) + 2x \\ &= \tfrac{6x^2}{61} + \tfrac{118x}{61} \end{aligned}$$

Moreover, since now we know $p_\infty(x)$, *we can corroborate*

$$B = p_\infty(x) \star \mathbf{P}_y(y) = p_\infty(x) \star g_1(y) = \int_0^1 \left(\frac{6 \cdot (1-y)^2}{61} + \frac{118 \cdot (1-y)}{61}\right) \cdot y^3 \, dy = \frac{6}{61}$$

via direct integration. A different way to calculate B *arises from noticing it is really the definition of* κ, *and we can calculate*

$$B = \kappa = \frac{\gamma}{1-\alpha} = \frac{\frac{1}{10}}{1+\frac{1}{60}} = \frac{6}{61}$$

11.6.2. *Other Eigenvalues and Eigenfunctions.*

Claim 11.40. ***(May 9, 2016)***

$$e(x) = A \cdot \mathbf{P}_x(x)$$

A is a constant, is the eigenfunction corresponding to eigenvalue $\lambda = \alpha$ *of*

$$p(x,y) = \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + f_n^*(x)$$

Proof The eigenfunction equation is suggestive of what we must do to prove the claim: $e(x) \star p(x,y) = \lambda e(x)$. We must show that, starring the eigenfunction with $p(x,y)$, we obtain α times the eigenfunction.

Thus:

$$\begin{aligned} e(x) \star p(x,y) &= A \cdot \mathbf{P}_x(x) \star (\mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + f_n^*(x)) \\ &= A \cdot \mathbf{P}_x(x) \star (\mathbf{P}_x(x) \cdot \mathbf{P}_y(y)) + \cancel{A \cdot \mathbf{P}_x(x) \star f_n^*(x)} \\ &= A \cdot \mathbf{P}_x(x) \cdot \underbrace{\mathbf{P}_x(x) \star \mathbf{P}_y(y)}_{\alpha} \\ &= \alpha \cdot A \cdot \mathbf{P}_x(x) \\ &= \alpha \cdot e(x) \end{aligned}$$

□

11.6.3. $c(x,0)$ *Transformations and Dynamics.* Provided a *Pasqualian* exists, of form

$$p(x,y,t) = \alpha^{t-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + f_n^*(x)$$

with $0 \le \alpha < 1$ or $0 \le \alpha \le 1 \wedge [\gamma = 0 \vee \mathbf{P}_x(x) \cdot \mathbf{1} = 0]$.

Claim 11.41. ***(July 11, 2016)*** *Take* $c(x,0)$, *a probability distribution, as always, in the interval* $[0,1]$. *Then*

$$c(x,t) = \delta \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) + f_n^*(x)$$

with $\delta = c(x,0) \star \mathbf{P}_y(y)$ *and* $0 \le \alpha \le 1$.

Proof This follows from the definition of $c(x,t) = c(x,0) \star p(x,y,t)$. Thus

$$\begin{aligned} c(x,t) &= c(x,0) \star \left(\alpha^{t-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y) + f_n^*(x)\right) \\ &= \alpha^{t-1} \cdot \mathbf{P}_x(x) \cdot c(x,0) \star \mathbf{P}_y(y) + c(x,0) \star f_n^*(x) \\ &= \delta \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) + f_n^*(x) \end{aligned}$$

□

Corollary 11.42. ***(July 11, 2016)*** *We have:*

$$\frac{\partial c(x,t)}{\partial t} = \delta \cdot \ln(\alpha) \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x)$$

In particular, the derivative stabilizes at infinity:

$$\lim_{t\to\infty} \frac{\partial c(x,t)}{\partial t} = 0$$

Proof We can approach this two ways, via direct derivation or formulaically.

The first approach implies

$$\begin{array}{rcl} \frac{\partial c(x,t)}{\partial t} & = & \frac{\partial}{\partial t}\left[\delta \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) + f_n^*(x)\right] \\ & = & \delta \cdot \frac{\partial}{\partial t}\left[\alpha^{t-1}\right] \cdot \mathbf{P}_x(x) \\ & = & \delta \cdot \ln(\alpha) \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) \end{array}$$

The second approach requires remembering that $\frac{\partial c(x,t)}{\partial t} = c(x,0) \star \frac{\partial p(x,y,t)}{\partial t}$:

$$\begin{array}{rcl} \frac{\partial c(x,t)}{\partial t} & = & c(x,0) \star \left[\ln(\alpha) \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) \cdot \mathbf{P}_y(y)\right] \\ & = & \ln(\alpha) \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) \cdot \overbrace{c(x,0) \star \mathbf{P}_y(y)}^{\delta} \\ & = & \delta \cdot \ln(\alpha) \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x) \end{array}$$

Having established such, let us calculate the limit as t goes to infinity:

$$\begin{array}{rcl} \lim_{t\to\infty} \frac{\partial c(x,t)}{\partial t} & = & \lim_{t\to\infty}\left[\ln(\alpha) \cdot \alpha^{t-1} \cdot \mathbf{P}_x(x)\right] \\ & = & \ln(\alpha) \cdot \lim_{t\to\infty}\left[\alpha^{t-1}\right] \cdot \mathbf{P}_x(x) \end{array}$$

which, for $0 \le \alpha < 1$, implies the claim via taking the limit of the exponentiation of α. For $\alpha = 1$, the result follows from $\ln(1) = 0$. □

11.6.4. *Orthogonality.* Again we assume the definitions of

$$e_1(x) = \kappa \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)$$

and

$$e_\alpha(x) = A \cdot \mathbf{P}_x(x)$$

Lemma 11.43. ***(July 19, 2016)***

$$\int_0^1 \mathbf{P}_x(x) \cdot e_\alpha(x)\, dx \ge 0$$

or

$$\int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx \ge 0$$

Proof This is easily shown, since

$$\int_0^1 \mathbf{P}_x(x) \cdot e_\alpha(x)\, dx = A \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx$$

Next, recall

$$\mathbf{P}_x(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$$

so that

$$\mathbf{P}_x(x) \cdot \mathbf{P}_x(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix} \cdot \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix} = \sum_{i=1}^{n} u_i^2(x)$$

with $u_i^2(x)\colon [0,1] \to \mathbb{R}^+ \cup \{0\}$ because each individual function's square is non-negative. Observe that the integral of each square function must be non-negative, so that:

$$\int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\,dx = \int_0^1 \sum_{i=1}^n u_i(x)\,dx = \sum_{i=1}^n \int_0^1 u_i(x)\,dx \geq 0$$

(we include the option of the integral equaling zero in the case all $u_i(x) = 0$ for all i. □

Claim 11.44. ***(July 19, 2016)*** *$e_1(x)$ and $e_\alpha(x)$ are orthogonal, that is,*

$$\int_0^1 e_1(x) \cdot e_\alpha(x)\,dx = 0$$

if and only if

$$\kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\,dx = -\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx$$

Proof By massaging the defining equation, we can prove the bidirectionality of the claim. Observe:

$$\begin{aligned} \textstyle\int_0^1 e_1(x) \cdot e_\alpha(x)\,dx &= \textstyle\int_0^1 (\kappa \cdot \mathbf{P}_x(x) \cdot \mathbf{1} + f_n^*(x)) \cdot A \cdot \mathbf{P}_x(x)\,dx \\ &= A \cdot \kappa \cdot \textstyle\int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\,dx + A \cdot \int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx \end{aligned}$$

so that

$$A \cdot \kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\,dx + A \cdot \int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx = 0$$

gives

$$\kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\,dx = -\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx$$

as desired.

□

Corollary 11.45. ***(July 20, 2016)***

$$\left[\kappa = 0 \iff \int_0^1 f_n^*(1-y) \cdot \mathbf{P}_y(y)\,dy = 0\right] \wedge \left[\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx = 0\right] \Rightarrow \int_0^1 e_1(x) \cdot e_\alpha(x)\,dx = 0$$

Proof *This is easy to see from* ***Claim 11.44****, specifically*

$$\kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\,dx = -\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx$$

From $\kappa = 0$ and orthogonaility of $f_n^*(x)$ and $\mathbf{P}_x(x)$ we obtain equality of the condition and therefore the orthogonality condition of the eigenfunctions.

Corollary 11.46. ***(July 20, 2016)***

$$[\exists Pasqualian] \wedge \left[\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx = 0\right] \Rightarrow \int_0^1 e_1(x) \cdot e_\alpha(x)\,dx = 0$$

Proof Since a *Pasqualian* exists, by **Claim 11.37** it follows $\kappa = 0$ and $0 \leq \alpha \leq 1$, so we can invoke **Corollary 11.45**. □

Corollary 11.47. ***(July 20, 2016)***

$$[\exists Pasqualian] \wedge [f_n^*(x) = 1] \Rightarrow \int_0^1 e_1(x) \cdot e_\alpha(x)\,dx = 0$$

Proof Since $f_n^*(x) = 1$, it follows that

$$\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx = \int_0^1 \mathbf{P}_x(x)\,dx = 0$$

Next, we can invoke **Corollary 11.46**. □

Corollary 11.48. ***(July 19, 2016)***

$$\left[\int_0^1 e_1(x) \cdot e_\alpha(x)\,dx = 0\right] \wedge \left[\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\,dx = 0\right] \Rightarrow \kappa = \gamma = 0$$

Proof This follows from **Claim 11.44**, since

$$\begin{array}{rcl} \kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx & = & -\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\, dx \\ & = & 0 \end{array}$$

Using **Lemma 11.43** and requiring $\mathbf{P}_x(x) \neq 0$, we have

$$\kappa \cdot \overbrace{\int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx}^{>0} = 0$$

and $\kappa = 0$. Since $\kappa = \frac{\gamma}{1-\alpha}$, it follows $\gamma = 0$. □

Corollary 11.49. ***(July 20, 2016)***

$$\left[\int_0^1 e_1(x) \cdot e_\alpha(x)\, dx = 0\right] \wedge [f_n^*(x) = 1] \Rightarrow \kappa = \gamma = 0 \Rightarrow \int_0^1 \mathbf{P}_y(y)\, dy = 0$$

Proof Again this is easily seen with **Claim 11.44**:

$$\begin{array}{rcl} \kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx & = & -\int_0^1 1 \cdot \mathbf{P}_x(x)\, dx \\ & = & 0 \end{array}$$

and **Lemma 11.43**, requiring $\mathbf{P}_x(x) \neq 0$, we, once more, have

$$\kappa \cdot \overbrace{\int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx}^{>0} = 0$$

and $\kappa = \gamma = 0$. Since $\gamma = f_n^*(x) \star \mathbf{P}_y(y)$, it follows $1 \star \mathbf{P}_y(y) = \int_0^1 \mathbf{P}_y(y)\, dy = 0$. □

Corollary 11.50. ***(August 1, 2016)*** *This is the last claim that refers to **Claim 11.44**, and, unremarkable as it is, we specify it for completeness.*

$$\left[\int_0^1 e_1(x) \cdot e_\alpha(x)\, dx = 0\right] \wedge [\kappa = 0] \Rightarrow \int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\, dx = 0$$

Proof Again, from **Claim 11.44** we have:

$$\cancel{\kappa \cdot \int_0^1 \mathbf{P}_x(x) \cdot \mathbf{P}_x(x)\, dx} = -\int_0^1 f_n^*(x) \cdot \mathbf{P}_x(x)\, dx$$

and the result follows. □

12. Other Claims and Proofs

12.1. **Combinatorics.** The following identity arose in my studies of the Mexican lottery.

Claim 12.1 (Elisa and Carlos Pasquali Combinatorial Identity)**.** ***(December 27, 2008)***

$$\binom{n-s}{r-s} \cdot \binom{n}{s} = \binom{n}{r} \cdot \binom{r}{s}$$

with $n \geq r \geq s \geq 0$

Proof By the definition of the choice operation,

$$\begin{aligned} \binom{n-s}{r-s} \cdot \binom{n}{s} &= \frac{(n-s)!}{(r-s)!(n-r)!} \cdot \frac{n!}{s!(n-s)!} \\ &= \frac{n!}{(r-s)!(n-r)!s!} \\ &= \frac{n!r!}{r!(n-r)!s!(r-s)!} \\ &= \frac{n!}{r!(n-r)!} \cdot \frac{r!}{s!(r-s)!} \\ &= \binom{n}{r} \cdot \binom{r}{s} \end{aligned}$$

□

12.2. **Markov Matrices.** The following claim arose due to my studies of Voting Theory and the Schulze method.

Claim 12.2. ***(March 2, 2011)*** *A $(n+p)\times(n+p)$ Markov matrix $M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, so that the upper left zero matrix is $n\times n$, A is $n\times p$, B is $p\times n$ and has the property that every entry is $\frac{1}{n}$, and the lower zero matrix is $p\times p$:*

1. *Has powers that are Markov matrices*
2. *Has positive even powers that are the same*
3. *Has positive odd powers that are the same, except possibly the first power*

Proof We have:

(1) Let M, N be $q\times q$ Markov matrices, so that

$$M = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q,1} & a_{q,2} & \cdots & a_{q,q} \end{bmatrix}$$

$$N = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,q} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q,1} & b_{q,2} & \cdots & b_{q,q} \end{bmatrix}$$

Then

$$M\cdot N = \begin{bmatrix} \sum_{j=1}^{q} a_{1,j}b_{j,1} & \sum_{j=1}^{q} a_{1,j}b_{j,2} & \cdots & \sum_{j=1}^{q} a_{1,j}b_{j,q} \\ \sum_{j=1}^{q} a_{2,j}b_{j,1} & \sum_{j=1}^{q} a_{2,j}b_{j,2} & \cdots & \sum_{j=1}^{q} a_{2,j}b_{j,q} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{q} a_{q,j}b_{j,1} & \sum_{j=1}^{q} a_{q,j}b_{j,2} & \cdots & \sum_{j=1}^{q} a_{q,j}b_{j,q} \end{bmatrix}$$

If we sum each row we have the vector

$$= \begin{bmatrix} \sum_{k=1}^{q}\sum_{j=1}^{q} a_{1,j}b_{j,k} \\ \sum_{k=1}^{q}\sum_{j=1}^{q} a_{2,j}b_{j,k} \\ \vdots \\ \sum_{k=1}^{q}\sum_{j=1}^{q} a_{q,j}b_{j,k} \end{bmatrix}$$

Since finite sums always converge, there is no issue exchanging the order of the sums (alternatively set $j=1$, show that $a_{1,1}$ can be factored and the remaining sum in b is one, and proceed by cycling through all j), so we have the vector:

$$= \begin{bmatrix} \sum_{j=1}^{q}\sum_{k=1}^{q} a_{1,j}b_{j,k} \\ \sum_{j=1}^{q}\sum_{k=1}^{q} a_{2,j}b_{j,k} \\ \vdots \\ \sum_{j=1}^{q}\sum_{k=1}^{q} a_{q,j}b_{j,k} \end{bmatrix}$$

or

$$= \begin{bmatrix} \sum_{j=1}^{q} a_{1,j}\sum_{k=1}^{q} b_{j,k} \\ \sum_{j=1}^{q} a_{2,j}\sum_{k=1}^{q} b_{j,k} \\ \vdots \\ \sum_{j=1}^{q} a_{q,j}\sum_{k=1}^{q} b_{j,k} \end{bmatrix}$$

For any value of j, $\sum_{k=1}^{q} b_{j,k} = 1$, so in effect we have

$$= \begin{bmatrix} \sum_{j=1}^{q} a_{1,j} \\ \sum_{j=1}^{q} a_{2,j} \\ \vdots \\ \sum_{j=1}^{q} a_{q,j} \end{bmatrix}$$

since we know the rows of M add up to one by hypothesis, we are left with a vector of size $q \times 1$ with entries all ones, as we wanted to show. Markov matrices are closed under matrix multiplication.

It follows that powers of a Markov matrix are Markov too, since we are multiplying Markov matrices by (the same) Markov matrix.

(2) The rather particular Markov matrix

$$M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

has second power:

$$M^2 = \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}$$

A remark: since M^2 is Markov, it follows that the $n \times n$ submatrix AB and the $p \times p$ submatrix BA are Markov too.

Now rewrite $B = \frac{1}{n}B^*$ so that $AB = A\frac{1}{n}B^*$ with B^* is a $p \times n$ matrix with entries all ones. Then AB^* adds the rows of A, which we know are equal to one, and $AB^* = C$ is a $n \times n$ matrix with entries all ones. Thus $AB = \frac{1}{n}C$.

Rewrite $BA = \frac{1}{n}B^*A$. Now let $B^*A = D$ is a $p \times p$ matrix with rows that are identical and sum the columns of A. As before we have that

$$M^2 = \frac{1}{n}\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$$

and

$$M^4 = \frac{1}{n^2}\begin{bmatrix} C^2 & 0 \\ 0 & D^2 \end{bmatrix}$$

Now, $C^2 = nC$ because of the property of C being all ones. We resort to a trick to show that $D^2 = nD$ as well, by writing out the explicit definition of D:

$$D = \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ a_1 & a_2 & \cdots & a_p \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_p \end{bmatrix}$$

and

$$D^2 = \begin{bmatrix} a_1 \sum_{i=1}^p a_i & a_2 \sum_{i=1}^p a_i & \cdots & a_p \sum_{i=1}^p a_i \\ \vdots & \vdots & \ddots & \vdots \\ a_1 \sum_{i=1}^p a_i & a_2 \sum_{i=1}^p a_i & \cdots & a_p \sum_{i=1}^p a_i \end{bmatrix}$$

or

$$D^2 = \left(\sum_{i=1}^p a_i\right) D$$

Careful, D is NOT Markov, so that the rows do not sum to 1, but D was generated summing the columns of A, a finite (non-Markov) matrix but with rows summing to 1, so that the sum operation in front of the D is asking us to do the double sum on the entries of A (sum all the entries of A). A is finite, so let's sum all the rows first and then all the columns (the order of the summing can be exchanged). Since all rows sum to 1, and there are n rows, it follows we are left with n as a result, and $D^2 = nD$.

[** From a patch-point-of-view, the statement that $D^2 = nD$, or $\left(\frac{1}{n}D\right)^2 = \frac{1}{n}D \Leftarrow N^2 = N$ with N is a Markov matrix with identical rows is analogous to the statement that $a(x) \star a(x) = a(x)$ with $a(x)$ is a *Pasquali patch*: at the "powering" level, the $\star$ operator as I defined it exchanges the integral summation and then adds continuously: the result of this is a uniform distribution (the analogue of the n in front of the D).**]

It is clear now that $M^4 = M^2$, and, as before, we use induction to show this is the case for all even powers of this particular matrix. So assume $M^{2k} = M^2$, and then $M^{2(k+1)} = M^{2k}M^2 = M^2M^2 = M^2$ and we are done.

(3) Since now we know that all even powers are the same, it follows that odd powers are the same too (except possibly the first power):

$$M^{2m+1} = M^{2m}M = M^2M = M^3$$

So all odd powers, except possibly M, look like M^3:

$$M^2M = \begin{bmatrix} \frac{1}{n}C & 0 \\ 0 & \frac{1}{n}D \end{bmatrix} \cdot \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{n}CA \\ \frac{1}{n}DB & 0 \end{bmatrix}$$

The product CA is a $(n \times n) \times (n \times p) = (n \times p)$ matrix with identical rows that adds the columns of A, call it D^*. On the other hand, the product $DB = D\frac{1}{n}B^*$ is a $(p \times p) \times (p \times n) = (p \times n)$ matrix that adds the rows of D which we already calculated must sum to n, so all its entries are such. Factor the n, and we get $DB = \frac{1}{n}nC^* = C^*$, with C^* all entries are ones. Finally, notice $\frac{1}{n}C^* = B$. We have:

$$M^3 = \begin{bmatrix} 0 & \frac{1}{n}D^* \\ B & 0 \end{bmatrix}$$

Since D^* was generated adding the columns of A (and A has entries that are nonzero at every row), one can think of several exceptions so that $\frac{1}{n}D^* \neq A$, and $M \neq M^3$. For example, in the special case where $n = p$, let A be the identity matrix; then $\frac{1}{n}D^*$ contains entries that are all $\frac{1}{n}$ and $\frac{1}{n}D^* \neq A \Rightarrow M \neq M^3$. □

Appendix

Appendix A. Proofs in Progress

Are convergent infinite sums of eigenvalues... descriptive of a finite function?

Can we invent new, finite-dot-product functions by looking at convergent infinite sums of eigenvalues?

Can we check equivalence of formulas by this method?

For the claim where even and odd derivatives are constrained, what happens to periodic functions?

Do they have the same invariant, sine and cosine? Sine and cosine plus a phase shift?

What is the relationship to the pythagorean theorem (can we prove it using this method)? Diophantine analysis with this method?

No reason why we should let surfaces be finite sums of x and y products.

Groups defined on patchixes or patches

All states are achievable

Alternative ways of writing the star operator (similarity to convolution), changing of variables to show equivalence

The derivative as progressive shape pathix changes using converging $\mathbb{P}$

In the diagram, the transversal iff

B of sin/cos examples (pasquali series)

Using complex numbers

Develop Entropy more

The pasqualian as a diffusion equation, similarity to shrodinger

Explore more the convergence criterion on sequences.

Corollary A.1. *Pick a vector (sub)space by fixing n so that $v \in \mathbb{R}^n$ has specification:*

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

and put the vectors in correspondence with elements $\left[\mathbf{\Omega}(\mathbb{Z}^+, \neq 0)\right]_{\bar{i}} \cup \omega_{[0]}(x)$ in the natural way, so that we may create

$$\omega_{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}(x)$$

Naturally, $\sum_i v_i \neq 0$ except for $\omega_{[0]}(x)$. Such correspondence is obviously one-to-one, and creates a vector subspace *of $\mathbb{R}^n$.*

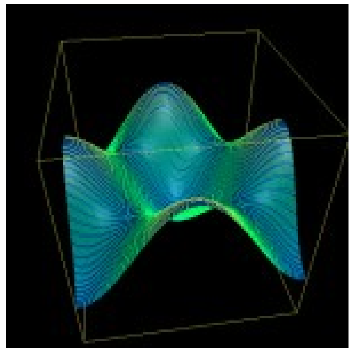


Figure 1. $p(x, y) = 1 - \cos(2\pi x)\cos(2\pi y)$

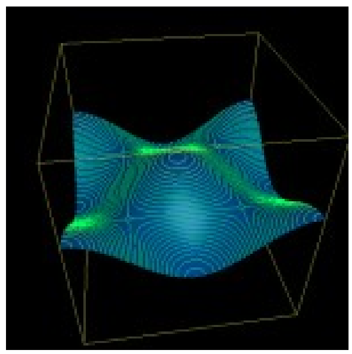


Figure 2. $p_2(x, y) = 1 + \frac{\cos(2\pi x)\cos(2\pi y)}{2}$

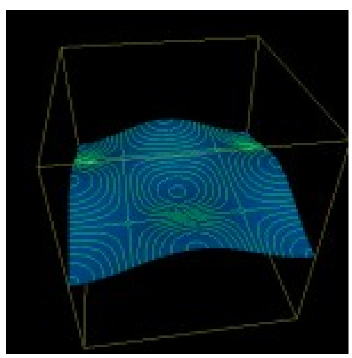


Figure 3. $p_3(x, y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{4}$

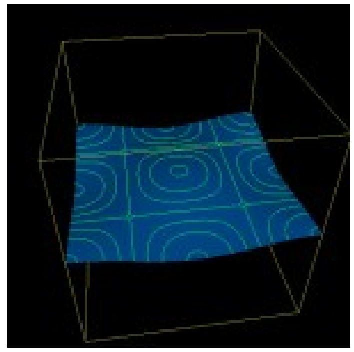


Figure 4. $p_4(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{8}$

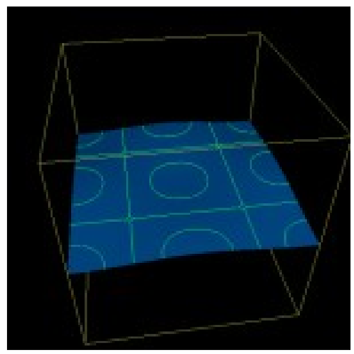


Figure 5. $p_5(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{16}$

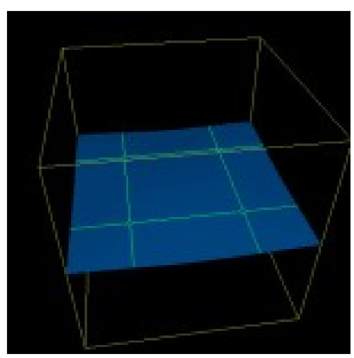


Figure 6. $p_6(x,y) = 1 - \frac{\cos(2\pi x)\cos(2\pi y)}{32}$

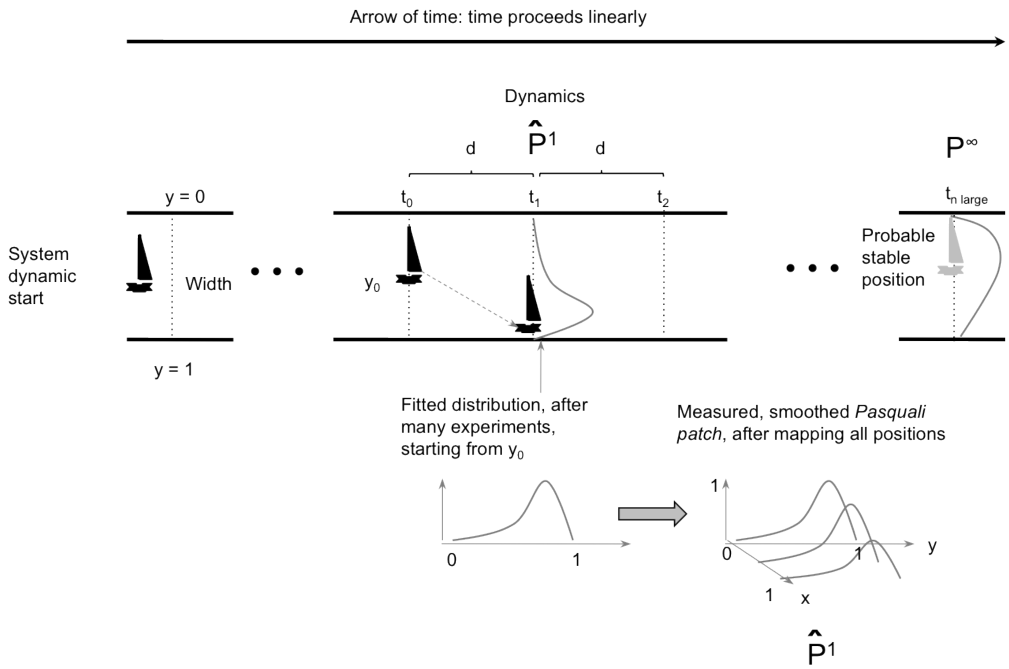


FIGURE 7. We can experimentally create a *Pasquali patch* and use it for prediction. We can perform the measurement at an arbitrary distance.

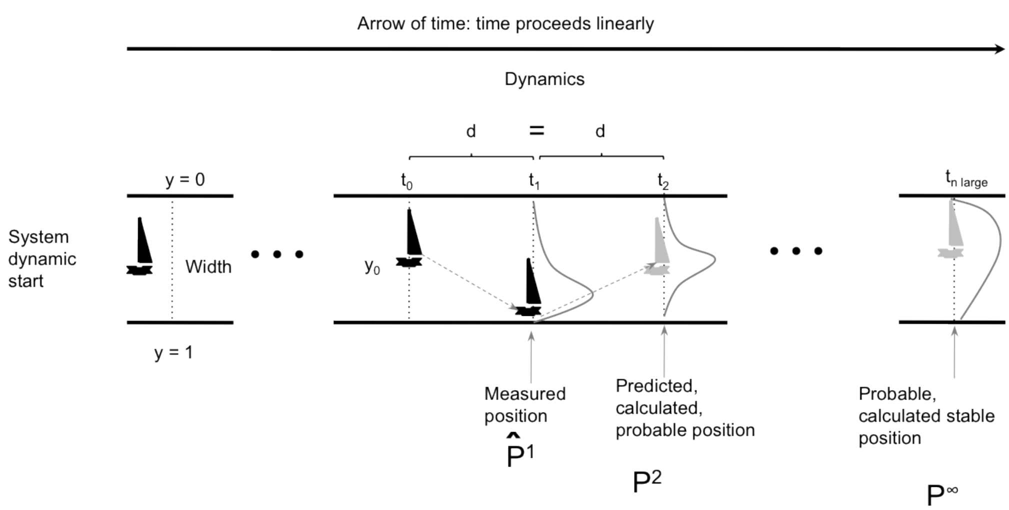


FIGURE 8. We can then use *Pasquali patch* powers for position prediction down the canal, at position $n \cdot \Delta t$ down the origin.

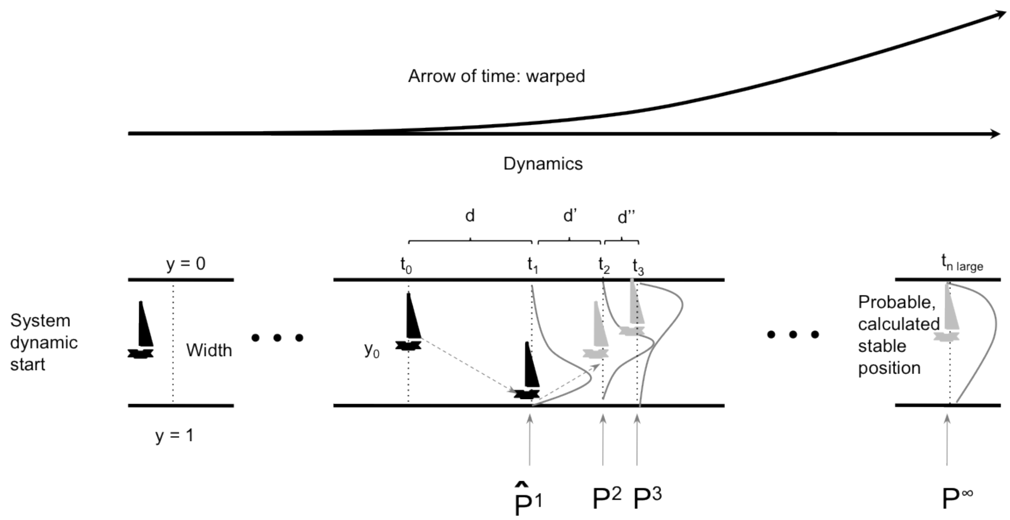


FIGURE 9. In this schematic a *Pasquali patch* and its powers do describe the system, but at non-equidistant points. The arrow of time is warped.

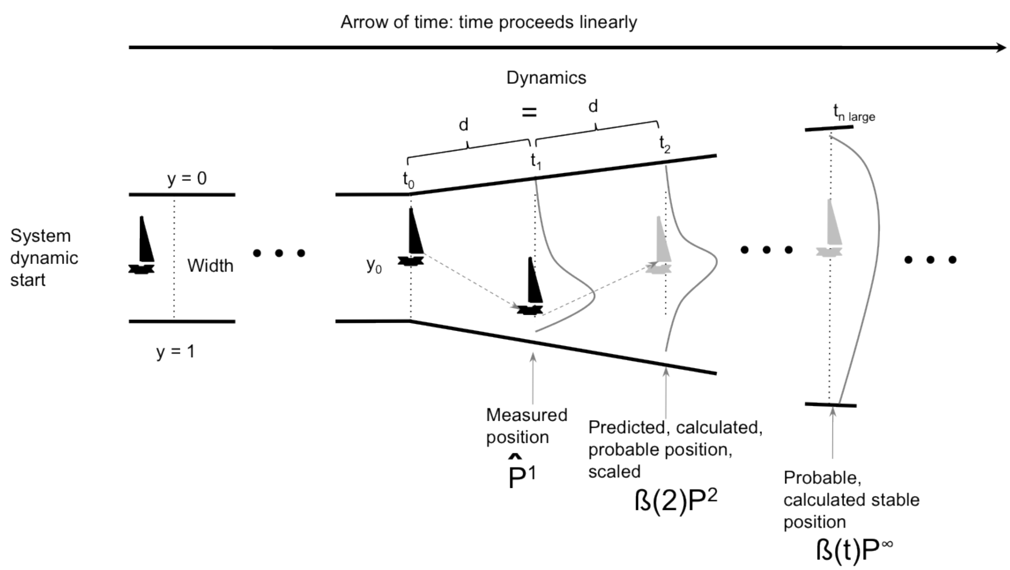


FIGURE 10. In this schematic the width of the canal grows linearly, but *Pasquali patch* dynamics are conserved, suitably scaled.